# A NOTE ON THE USE OF MEDIAN RANGES

BY MASAAKI SIBUYA

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In setting up quality control charts, the mean-range is often used to estimate the population standard deviation. E. B. Ferrel [1] has pointed out the possibility of using, instead of the mean-range, the median-range which is efficient enough for that purpose. In this paper Ferrel has given a table of expected values of the median-range for different values of $n$, the sample size. These are asymptotic values which are obtained under the assumption that the number, $N$, of ranges from which the median-range is obtained, is very large. In fact, they are the 50% points of the distributions of ranges from normal samples. These values also appear in [2]. In this note we examine an asymptotic formula for the expected value of the median-range in normal samples.

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In general, the expected value of the median $\tilde{X}$ in a sample of size $N$ from the population with probability density function $f(x)$ is shown (see [3]) to be

$$(1) \qquad E(\tilde{X})=x_{0.5}-\frac{1}{8(N+2)}\,\frac{f'(x_{0.5})}{f^3(x_{0.5})}+O\left(\frac{1}{N^2}\right), \qquad \text{when } N \text{ is odd,}$$

and

$$(2) \qquad E(\tilde{X})=x_{0.5}-\frac{1}{8(N+1)}\,\frac{f'(x_{0.5})}{f^3(x_{0.5})}+O\left(\frac{1}{N^2}\right), \qquad \text{when } N \text{ is even,}$$

where $x_{0.5}$ is the 50% point of the distribution.

From the tables of the probability density function of the range [4], $f(x_{0.5})$ have been computed by interpolation and $f'(x_{0.5})$ by numerical differentiation.

Let $R$ be the median of $N$ ranges in independent normal samples of size $n$.

Then

$$(3) \qquad \frac{E(\tilde{R})}{\sigma}=\begin{cases} d_m+\dfrac{e}{N+2}+O\left(\dfrac{1}{N^2}\right), & \text{when } N \text{ is odd,} \\ d_m+\dfrac{e}{N+1}+O\left(\dfrac{1}{N^2}\right), & \text{when } N \text{ is even,} \end{cases}$$

where $d_m$ and $e$ are function of $n$, values of which are shown in the following table 1.

In setting up quality control charts, the second term which is small may be neglected.

Table 1

| $n$ | $d_m$ | $e$ |
|---|---|---|
| 2 | 0.95387 | 0.29519 |
| 3 | 1.588 | 0.162 |
| 4 | 1.978 | 0.124 |
| 5 | 2.257 | 0.108 |
| 6 | 2.472 | 0.098 |
| 7 | 2.645 | 0.093 |
| 8 | 2.791 | 0.090 |
| 9 | 2.915 | 0.086 |
| 10 | 3.024 | 0.084 |

In case $n=2$, the distribution function of normal range is

$$F(R)=2\Phi\left(\frac{R}{\sqrt{2}\,\sigma}\right)-1, \qquad 0<R<\infty, \tag{4}$$

where $\Phi(x)$ is the distribution function of standard normal distribution. Further, if $N$ is odd, i.e., $N=2M+1$,

$$\frac{E(\tilde{R})}{\sigma}=\frac{2\sqrt{2}N!}{(M!)^2}\int_0^\infty x[2\Phi(x)-1]^M[2-2\Phi(x)]^M d\Phi(x). \tag{5}$$

The values of (5) were computed for $N=3$ (2) 15 by numerical integration, the results of which are shown in table 2 with the approximate values based on (3). The variances and the efficiency of $\tilde{R}$ relative to the mean-range $\bar{R}$ in the estimation of $\sigma$ were also computed at the same time.

Table 2

| $N$ | $E(\tilde{R})/\sigma$ Exact | $E(\tilde{R})/\sigma$ Approx. | $V(\tilde{R})/\sigma^2$ | $V(\bar{R}/E(R))$ $/V(\tilde{R}/E(\tilde{R}))$ |
|---|---|---|---|---|
| 3 | 1.03572 | 1.013 | 0.33637 | 0.6068 |
| 5 | 1.00685 | 0.996 | 0.21807 | 0.5306 |
| 7 | 0.99295 | 0.987 | 0.16128 | 0.4985 |
| 9 | 0.98481 | 0.981 | 0.12794 | 0.4808 |
| 11 | 0.97946 | 0.977 | 0.10603 | 0.4695 |
| 13 | 0.97569 | 0.974 | 0.09052 | 0.4618 |
| 15 | 0.97289 | 0.9712 | 0.07897 | 0.4561 |
| 17 | 0.97072 | 0.9694 | 0.07003 | 0.4518 |

THE INSTITUTE OF STATISTICAL MATHEMATICS

## REFERENCES

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