ON THE STATISTICAL ESTIMATION OF FREQUENCY RESPONSE FUNCTION

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1. Introduction and summary

At present, the spectral method is used very commonly for the analysis of an electrical or mechanical system. The spectral method is used not only for the estimation of the individual spectral density functions of the input and output of the system but also for the estimation of the frequency characteristics of the system.

The statistical method of estimation of the frequency characteristics of a system is best suited for this purpose as it can be applied without disturbing the normal operation of the system and even under the existence of the additive disturbance of extraneous noises. Statistical method for the estimation of the power spectral density has been brought to a considerable development by the valuable contributions of many statisticians, for example, those of J. W. Tukey [4, 9]. As for the estimation of the frequency response function of a linear and time-invariant system we have a paper by N. R. Goodman [8]. The method described by Goodman was a direct application of the method of estimation of the spectral density to that of the crosspectral density, but some experimenters who applied this kind of method to their numerical data experienced the very low coherency of their estimates. As far as we know, this fact was first recognized experimentally and announced by J. F. Darzell and Y. Yamanouchi [6].

The fruitful results of the method of estimation of the spectral density are mainly due to its success in reducing the variance by using proper smoothing operations. Methods of smoothing or averaging such as those named "hanning" and "hamming" and so forth were derived as those which have desirable properties of concentration of the effective range of smoothing. The smoothing operation is carried out by taking the product of the sample covariance function and a smoothing kernel or a lag window. It is well known that the autocovariance function does not contain any information about the phase of each frequency component contained in the original data. However, the crosscovariance function contains information of the phase, and the alignment of the phase shift of the frequency response function at frequencies in the effective range of the smoothing is most important to get a valid estimate of the amplitude gain. In Goodman's paper little attention was paid to this fact. Some experimenters who applied the same kind of method directly to their numerical data observed a very low coherency of their estimates at frequencies near the peak of the amplitude gain.

In 1960 we gave a proof that this low coherency is due to the inappropriate use of the lag window [3] and showed that, in the case of Darzell and Yamanouchi [6], the shifting of the time axis of one of the data greatly improves the coherency [11]. Almost at the same time Darzell and W. J. Pierson Jr. published an interesting paper in which the proper use of the time shift operation was also recommended to avoid the reduction in coherency [7]. Because of the reasons stated in the preceding paper [1, p. 2], discussions in this paper will be restricted to the lag windows of trigonometric sum type which are defined in section 2. First the sampling variability of the estimate of a frequency response function obtained by using a smoothing operation will be evaluated and then the bias caused by use of a lag window will be analysed. Even if the amplitude gain of the system under investigation is assumed to be locally constant or linear in the range of the smoothing operation, rapid change of the phase shift may still occur, and it is shown that this tends to reduce the value of the estimate of the amplitude gain. When we recall that in the physically realizable minimum-phase system the rate of change of the phase shift with respect to the increase of frequency is large at the peak of the amplitude gain [10, p. 45], we can see that this observation is of general meaning. Further the bias due to the use of the window is observed to be usually small in the case of the estimation of the phase shift. Taking into account that for the input of white noise the overall reduction in coherency is kept minimum when the integral of the square of a crosscovariance function multiplied by the lag window is maximum, it is recommended, to obtain a good overall view, in the practical application of the present estimation procedure, first to shift the time axis of the crosscovariance function so as to situate the origin at the maximum of the envelope of the sample crosscovariance function. The result of the analysis of the sampling variability of the estimate shows that at frequencies where the values of the coherency are high the lag window of which the Fourier transform, or the spectral window, has narrow bandwidth should be used. Numerical comparison of the windows are made and the use of the window Q, which has been recommended for estimation of the power spectral density in the preceding paper, is recommended for this case too. Sampling variability of an estimate of the coherency is discussed and the mean and the variance of the estimate are numerically given for some typical cases. Formulae for approximate evaluation of these two quantities are also given. Construction of the confidence region for the frequency response function is tentatively suggested. In the final section some numerical examples are given to show a practical meaning of the contents of this paper.

2. Sampling variability of the estimate

In this section an estimation procedure of the frequency response function of a linear time-invariant system will be discussed and the sampling variability of the estimate will be evaluated. The evaluation proceeds entirely analogously to that of the estimate of the spectral density function treated in the preceding paper [1]. All the stochastic processes treated in this parer are assumed to be Gaussian. Therefore the results of the evaluation of the sampling variability of the estimate have only the meaning of rough approximation to non-Gaussian cases. Even for the Gaussian case the evaluation can be carried out only approximately, so in practical applications of the method it is most recommended to check the sampling variability of the estimate numerically by some artificial experiments.

We observe here two stationary stochastic processes $x(t)$ and $y(t)$ which are real and are related by

$$x(t)=\int_{-\infty}^{\infty} \exp(2\pi i f t) dZ(f)^*$$

$$y(t)=\int_{-\infty}^{\infty} \exp(2\pi i f t) A(f) dZ(f)+n(t)$$

where $A(f)$ is the frequency response function or the gain of the system under consideration and $n(t)$ is a Gaussian process which is independent of the process $x(t)$, and which represents the additive disturbance of the extraneous noise. $x(t)$ and $n(t)$ are assumed to have absolutely continuous power spectral distributions with power spectral density functions $p_x(f)$ and $p_n(f)$ respectively, and to have zero-means.

We shall assume $p_x(f)$, $p_n(f)$ and $|A(f)|$ to be bounded continuous functions of f.

Under the present assumption the process $y(t)$ has an absolutely continuous power spectral distribution with power spectral density $p_y(f)$ given by

$$p_y(f)=|A(f)|^2 p_x(f)+p_n(f).$$

We shall also use the Fourier representation of $n(t)$

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* This is the Fourier representation of $x(t)$ and $Z(f)$ is a complex orthogonal process with $E|dZ(f)|^2=p_x(f)df$.

$$n(t)=\int_{-\infty}^{\infty}\exp(2\pi ift)dN(f).$$

It is known that when $y(t)$ represents a stationary output of a linear time-invariant system, of which the impulsive response function is $h(t)$, and which is distorted with an additive random noise $n(t)$, i.e.,

$$y(t)=\int_{0}^{\infty}x(t-\tau)h(\tau)d\tau+n(t),$$

we have

$$A(f)=\int_{0}^{\infty}\exp(-2\pi if\tau)h(\tau)d\tau.$$

$|A(f)|$ is called the amplitude gain of the system and $\phi(f)=\arg A(f)$ the phase shift of the system. Now, suppose data are given and are represented by

$$\{(x(t),\, y(t));\, -T\leqq t\leqq T\}.$$

After the preceding paper [1, p. 3] we define for integral ν's

$$X\left(\frac{\nu}{2T}\right)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}t\right)x(t)dt$$

$$Y\left(\frac{\nu}{2T}\right)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}t\right)y(t)dt$$

$$N\left(\frac{\nu}{2T}\right)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}t\right)n(t)dt.$$

Then

$$X\left(\frac{\nu}{2T}\right)=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dZ(f)$$

$$Y\left(\frac{\nu}{2T}\right)=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)A(f)dZ(f)+N\left(\frac{\nu}{2T}\right)$$

$$N\left(\frac{\nu}{2T}\right)=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dN(f)$$

where

$$W_T(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(-2\pi ift)dt.$$

Throughout the present paper it is assumed that T is sufficiently large so that, the real and the imaginary parts of $X(\nu/2T)$'s or $N(\mu/2T)$'s can be considered to be mutually independent Gaussian random variables with zero means and variances $p_x(\nu/2T)/2$ or $p_n(\mu/2T)/2$, respectively, for positive ν and μ, as was observed in the preceding paper, [1. p. 5].

Obviously, from the assumption of this paper it follows that $X(\nu/2T)$'s and $N(\mu/2T)$'s are mutually independent.

Hereafter, in this section, we shall adopt the extremely simplified assumption of constant $A(f)$. Then

$$\int_{-\infty}^{\infty} W_T\left(\frac{\nu}{2T}-f\right)A(f)dZ(f)\approx A\left(\frac{\nu}{2T}\right)\int_{-\infty}^{\infty} W_T\left(\frac{\nu}{2T}-f\right)dZ(f)$$

and

$$Y\left(\frac{\nu}{2T}\right)\overline{X\left(\frac{\nu}{2T}\right)}\approx A\left(\frac{\nu}{2T}\right)\left|X\left(\frac{\nu}{2T}\right)\right|^2+N\left(\frac{\nu}{2T}\right)\overline{X\left(\frac{\nu}{2T}\right)}$$

where $\approx$ means "equal to" under the present assumption. From this result one may imagine that the statistic

$$\frac{Y\left(\frac{\nu}{2T}\right)\overline{X\left(\frac{\nu}{2T}\right)}}{\left|X\left(\frac{\nu}{2T}\right)\right|^2}\approx A\left(\frac{\nu}{2T}\right)+\frac{N\left(\frac{\nu}{2T}\right)}{X\left(\frac{\nu}{2T}\right)}$$

will form an estimate of $A(\nu/2T)$. However, taking into account that $(p_x(\nu/2T))^{-1}|X(\nu/2T)|^2$ is distributed approximately as a χ^2 with d.f. 2, we can see that the square of the absolute value of the error term $X(\nu/2T)^{-1}N(\nu/2T)$ will not have finite expectation. This fact suggests that the statistic of this type will not be an appropriate one for the estimation of $A(f)$, and that it will be necessary to increase the d.f. of the denominator to obtain a reliable estimate. Using proper weight $\{w_\nu\}$ (satisfying $\sum_\nu w_\nu=1$) we now introduce our estimate $A(f_\mu)$:

$$\hat{A}(f_\mu)=\frac{\sum_\nu w_\nu Y(f_\mu-f_\nu)\overline{X(f_\mu-f_\nu)}}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}\ {}^*$$

where $f_\mu=\mu/2T$. For this estimate we have

$$\hat{A}(f_\mu)\approx A(f_\mu)+\frac{\sum_\nu w_\nu N(f_\mu-f_\nu)\overline{X(f_\mu-f_\nu)}}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}.$$

Taking into account that we have almost surely $\sum_\nu|X(f_\nu)|^2=\int_{-T}^{T}|x(t)|^2dt$ and $\sum_\nu|N(f_\nu)|^2=\int_{-T}^{T}|n(t)|^2dt$, we get, in case w_ν's $(\nu\geqq\mu)$ are small enough and the last quantity in the following is finite,

*In this paper we shall use the notation $\sum_\nu$ in place of $\sum_{\nu=-\infty}^{\infty}$ and $\sum_{\nu\neq\mu}$ in place of $\sum_{\nu=-\infty}^{\mu-1}+\sum_{\nu=\mu+1}^{\infty}$.

$$E\left|\frac{\sum_\nu w_\nu N(f_\mu-f_\nu)\overline{X(f_\mu-f_\nu)}}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}\right|^2$$

$$=\sum_\nu\sum_{\nu'} w_\nu w_{\nu'} E\left[\frac{N(f_\mu-f_\nu)\overline{N(f_\mu-f_{\nu'})}\,\overline{X(f_\mu-f_\nu)}X(f_\mu-f_{\nu'})}{|\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2|^2}\right]$$

$$\sim E\left[\frac{\sum_\nu |w_\nu|^2|X(f_\mu-f_\nu)|^2}{|\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2|^2}\right]p_n(f_\mu)$$

$$\leqq \underset{\nu'}{\mathrm{Max}}|w_{\nu'}|E\left[\frac{\sum_\nu |w_\nu||X(f_\mu-f_\nu)|^2}{|\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2|^2}\right]p_n(f_\mu)$$

where the symbol $\sim$ means "approximately equal to" when T is sufficiently large so that $p_n(f_\mu-f_\nu)$ can be considered to be a constant $p_n(f_\mu)$ in evaluating the quantity under consideration. For almost all the lag windows which will be used practially it holds that $\underset{\nu'}{\mathrm{Max}}|w_{\nu'}|=w_0$ and $w_\nu=|w_\nu|$ at those ν's where $|w_\nu|$ is large. Therefore, in this case, the last term may be replaced by

$$w_0 E\left[\frac{1}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}\right]p_n(f_\mu).$$

In this paper we approximately regard the distribution of variable $\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2 p_x(f_\mu)^{-1}$ as a Γ-distribution, or more roughly as a χ^2-distribution, of which characteristics are given by $E(\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)$ and $D^2(\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)$. If a χ^2-distribution is adopted, its d.f. h is given by

$$h=\text{the nearest integer to }\frac{2(E\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)^2}{D^2(\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)}\sim\frac{2}{\sum_\nu|w_\nu|^2},$$

and when $h>2$

$$E\left[\frac{1}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}\right]\sim\frac{h}{h-2}\,p_x(f_\mu)^{-1}.$$

Thus an evaluation formula of the magnitude of the relative error of our estimate $\hat{A}(f_\mu)$ is given by

$$E\left|\frac{\hat{A}(f_\mu)-A(f_\mu)}{A(f_\mu)}\right|^2\sim w_0\frac{h}{h-2}\,\frac{p_n(f_\mu)}{|A(f_\mu)|^2 p_x(f_\mu)}\,.$$

This evaluation may seem to be somewhat conservative. However, when we compare numerically the present result with a more optimistic one, which is obtained by assuming $\sum_\nu |w_\nu|^2 |X(f_\mu - f_\nu)|^2 / \sum_\nu w_\nu |X(f_\mu - f_\nu)|^2 \doteqdot \sum_\nu w_\nu^2 / \sum_\nu w_\nu$ and is given by replacing w_0 in the above formula by $\sum_\nu w_\nu^2$, we can see that the difference is small, say of the order of 10 to 20% in the sense of root mean square, for the windows of ordinary use. From this result we may conclude that we can conveniently use any of these two formulae for the evaluation and analysis of the sampling variability of our estimate.

Following the definition in the preceding paper [1] let us call the weight $\{w_\nu\}$ a spectral window corresponding to a lag window $W(t)$ when it satisfies

$$w_\nu = \frac{1}{2T} \int_{-T}^{T} \exp\left(-2\pi i \frac{\nu}{2T} t\right) W(t) dt.$$

The lag window of trigonometric sum type of the kth order is defined by

$$\begin{aligned} W(t) &= \sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m} t\right) && |t| < T_m{}^* \\ &= \frac{1}{2} \sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m} t\right) && |t| = T_m \\ &= 0 && |t| > T_m \end{aligned}$$

where a_n's are real and $a_{-n} = a_n$.

For the lag windows of trigonometric sum type

$$w_0 = \frac{1}{2T} \int_{-T}^{T} W(t) dt = \frac{2T_m}{2T} a_0$$

$$\sum_n |w_n|^2 = \frac{1}{2T} \int_{-T}^{T} |W(t)|^2 dt = \frac{2T_m}{2T} \sum_{n=-k}^{k} |a_n|^2$$

and

$$h = \text{the nearest integer to } 2\left(\frac{2T}{T_m}\right) \frac{1}{2 \sum_{n=-k}^{k} |a_n|^2}.$$

The coherency $\gamma^2(f)$ at the frequency f is defined by

$$\begin{aligned} \gamma^2(f) &= \frac{|A(f)|^2 p_x(f)}{p_y(f)} \\ &= 1 - \frac{p_n(f)}{p_y(f)}. \end{aligned}$$

* If $\sum_\nu |w_\nu| < +\infty$ is necessary, we have only to modify this $W(t)$ to be smooth at $t = \pm T_m$. See also [1, p. 9].

When we assume that h is fairly large so that $h(h-2)^{-1}$ can be taken to be nearly equal to 1, the final evaluation formula of the relative accuracy of our estimate of the frequency response function will be given by

$$E\left|\frac{\hat{A}(f_\mu)-A(f_\mu)}{A(f_\mu)}\right|^2 \sim \frac{T_m}{2T} 2a_0 \frac{1-\gamma^2(f_\mu)}{\gamma^2(f_\mu)} .$$

If the above stated second evaluation formula is adopted,

$$E\left|\frac{\hat{A}(f_\mu)-A(f_\mu)}{A(f_\mu)}\right|^2 \sim \frac{T_m}{2T} 2 \sum_{n=-k}^{k} a_n^2 \frac{1-\gamma^2(f_\mu)}{\gamma^2(f_\mu)} .$$

By using this result and that of the analysis of bias, which will be given in the following section 3, the design principle of the optimum lag window of trigonometric sum type may be given entirely analogously as in the preceding paper. Therefore, we can see that the results of the analysis of the windows obtained in the preceding paper are all useful for this case too.

We have derived the present evaluation formulae under the assumption that $A(f)$ can be considered to be a constant in the range of the smoothing. The assumption of this kind caused little trouble in the case of estimation of the power spectral density, but in the case of estimation of the frequency response function the situation is different.

The lag windows of trigonometric sum type that have been used hitherto are all symmetric with respect to the y-axis hence are free from the biases of odd orders and they caused little trouble in the estimation of the power spectral density even in the case where the density function showed large linear variation. As to the estimation at the peaks and valleys of the power spectral density function, statisticians say that they are estimating the "averaged power spectra". In the estimation of the frequency response function the situation is the same and one must satisfy himself with an estimate of the "averaged frequency response function". However, there exists a difference in that the total power remains unchanged by averaging while the frequency response function may even "disappear" by averaging when the variation of phase shift exists. The apparent reduction in the amplitude gain is thus introduced by averaging and in case where the main concern of the experimenter is in the analysis of the coherency or of the linearity of the system such an estimate may be quite useless or even misleading.

In the following section the bias introduced by averaging in the estimation of the frequency response function will be discussed.

3. Bias due to smoothing

In this section we adopt the approximation to regard

$$Y(f_\mu)=A(f_\nu)X(f_\mu)+N(f_\mu) \qquad \mu=0, \pm 1, \pm 2, \cdots$$

which will hold strictly only when the $x(t)$ is circulating with period $2T$ and will hold approximately in the sense of mean square, as T tends to infinity. Then assuming the existence of $E[\sum_\nu |w_\nu||X(f_\mu-f_\nu)|^2/\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2]$ we have

$$E(\hat{A}(f_\mu))\sim \sum_\nu A(f_\mu-f_\nu)E\left[\frac{w_\nu|X(f_\mu-f_\nu)|^2}{\sum_\lambda w_\lambda|X(f_\mu-f_\lambda)|^2}\right].$$

For the sake of simplicity put $x=w_\nu|X(f_\mu-f_\nu)|^2$ and $y=\sum_{\lambda\neq\nu} w_\lambda|X(f_\mu-f_\lambda)|^2$. Even under present assumption that $|X(f_\mu-f_\nu)|$'s are mutually independent and follow distributions of χ^2-type with d.f. 2, the quantity $x/(x+y)$ may not have any finite expectation when some of the w_λ's take negative values. Obviously this is due to the positive probability of $x+y$ being in the neighbourhood of zero, and if this probability is very small we may practically restrict our consideration to the case where $x+y$ is greater than some positive constant. For evaluation of the expectation of $x/(x+y)$ in this restricted sense we shall here approximate $x/(x+y)$ by $(x/(x+y))_\Delta$ which is defined as

$$\left(\frac{x}{x+y}\right)_\Delta=\frac{1}{x_0+y_0}(x_0+\Delta x)\left(1-\frac{\Delta x+\Delta y}{x_0+y_0}+\left(\frac{\Delta x+\Delta y}{x_0+y_0}\right)^2\right)$$

where $x_0=Ex$, $y_0=Ey$, $\Delta x=x-x_0$ add $\Delta y=y-y_0$. When the coefficient of variation of $x+y$, which is approximately equal to $\sqrt{2/h}$, is, fairly small, say less than 1/3, this approximation will give an estimate of the bias of $x/(x+y)$ from $x_0/(x_0+y_0)$.

The expectation of $(x/(x+y))_\Delta$ is

$$E\left(\frac{x}{x+y}\right)_\Delta=\frac{x_0}{x_0+y_0}-\frac{E(\Delta x)^2}{(x_0+y_0)^2}+\frac{x_0E(\Delta x+\Delta y)^2}{(x_0+y_0)^3}+\frac{E(\Delta x)^3}{(x_0+y_0)^3}.$$

Under the assumption of this paper

$$x_0=w_\nu p_x(f_\mu-f_\nu)$$
$$x_0+y_0=\sum_\lambda w_\lambda p_x(f_\mu-f_\lambda)$$
$$E(\Delta x)^2=w_\nu^2p_x^2(f_\mu-f_\nu)$$
$$E(\Delta x+\Delta y)^2=\sum_\lambda w_\lambda^2p_x^2(f_\mu-f_\lambda)$$
$$E(\Delta x)^3=2(w_\nu p_x(f_\mu-f_\nu))^3$$

when $p_x(f)$ is considered to be nearly a constant in the neighourhood of f_μ, and it can be seen that the ratio of the last term of the right hand side of $E(x/(x+y))_\Delta$ to the third term is nearly equal to $2w_\nu^2/\sum_\lambda w_\lambda^2$. Taking into account the relations

$$w_0=\frac{2T_m}{2T}a_0$$

$$\sum_\lambda w_\lambda^2=\frac{2T_m}{2T}\sum_n a_n^2$$

for the lag windows of trigonometric sum type, we can see that when h defined in section 2 or T/T_m is fairly large this last term usually takes a very small value for the windws used ordinarily. For the evaluation, therefore $\bar{E}(x/(x+y))_\Delta$ will be used in place of $E(x/(x+y))_\Delta$ where $\bar{E}(x/(x+y))_\Delta$ is defined as

$$\bar{E}\left(\frac{x}{x+y}\right)_\Delta=\frac{x_0}{x_0+y_0}\left(1+\frac{E(\Delta x+\Delta y)^2}{(x_0+y_0)^2}\right)-\frac{E(\Delta x)^2}{(x_0+y_0)^2}\,.$$

This $\bar{E}(x/(x+y))_\Delta$ gives the accurate result of $E(\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2/\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)=1$ when it is formully applied to the evaluation of individual $E(w_\mu|X(f_\mu-f_\nu)|^2/\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2)$, which is a main reason why we have adopted this $\bar{E}(x/(x+y))_\Delta$ in the present investigation.*

When we assume that $p_x(f)$ is locally flat around f_μ we have

$$E(\hat{A}(f_\mu))=\sum_\nu A(f_\mu-f_\nu)\{w_\nu(1+\sum w_\lambda^2)-w_\nu^2\}$$

where the expectation should be understood in the restricted sense as mentioned above. When the assumption of local flatness of $p_x(f)$ is not satisfied, we have only to replace w_λ's in this formula of $E(\hat{A}(f_\mu))$ by w'_λ's which are proportional to $w_\lambda p_x(f_\mu-f_\lambda)$.

We analyse here the effect of smoothing operation when the assumption of constant $A(f_\mu-f_\nu)$ is not admitted. Besides the bias of $E(\hat{A}(f_\mu))$ from $A(f_\mu)$, the variation of $A(f_\mu-f_\nu)$ causes the sampling variability of the term $\sum_\nu w_\nu A(f_\mu-f_\nu)|X(f_\mu-f_\nu)|^2/\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2$. However, taking into account that there usually exists inevitable sampling variability due to the additive noise, we shall limit our attension to the analysis of the bias of $E(\hat{A}(f_\mu))$ from $A(f_\mu)$ assuming that $p_x(f)$ is locally flat around f_μ. It should be noted that when $A(f)$ can be con-

* Strict evaluation of $E(x/(x+y))$ is possible if $w_\lambda>0$ and $w_\lambda p_x(f_\mu-f_\lambda)\neq w_{\lambda'}p_x(f_\mu-f_{\lambda'})$ $(\lambda\neq\lambda')$ hold, but the result is not directly applicable to the present analysis.

sidered to be a constant in the effective range of the smoothing operation the assumption of local flatness of $p_x(f)$ is unnecessary to keep our estimate unbiased. This shows that the local flatness of the amplitude gain and that of phase shift have great influence on the estimation of the frequency response function. Therefore, we can guess that in this case some sort of "preflatening" operation will play the important role which was played by the prewhitening operation in the estimation of the power spectral density function.

Now let us consider the case where $A(f)$ admits the local approximation

$$A(f_\mu - f_\nu) = \alpha(1+\beta f_\nu + \gamma f_\nu^2)\exp(2\pi i f_\nu T_\mu)^*$$

where β, γ and T_μ are real constants and $\alpha = A(f_\mu)$. Then

$$E(\hat{A}(f_\mu)) \sim \sum_\nu A(f_\mu - f_\nu)\{w_\nu(1+\sum_\lambda w_\lambda^2) - w_\nu^2\}$$

$$= \alpha(\sum_\nu w_\nu \exp(2\pi i f_\nu T_\mu) + \beta \sum_\nu w_\nu f_\nu \exp(2\pi i f_\nu T_\mu)$$

$$+ \gamma \sum_\nu w_\nu f_\nu^2 \exp(2\pi i f_\nu T_\mu))(1+\sum_\lambda w_\lambda^2)$$

$$-\alpha(\sum_\nu w_\nu^2 \exp(2\pi i f_\nu T_\mu) + \beta \sum_\nu w_\nu^2 f_\nu \exp(2\pi i f_\nu T_\mu) + \gamma \sum_\nu w_\nu^2 f_\nu^2 \exp(2\pi i f_\nu T_\mu))$$

$$= \alpha(W(T_\mu) + \frac{\beta}{2\pi i}\dot{W}(T_\mu) + \frac{\gamma}{(2\pi i)^2}\ddot{W}(T_\mu))$$

$$+ (\sum_\lambda w_\lambda^2)\alpha\left\{W(T_\mu) - W^{*2}(T_\mu) + \frac{\beta}{2\mu i}(\dot{W}(T_\mu) - \dot{W}^{*2}(T_\mu))\right.$$

$$\left. + \frac{\gamma}{(2\pi i)^2}(\ddot{W}(T_\mu) - \ddot{W}^{*2}(T_\mu))\right\} (C.\ 3)$$

where $(C.\ 3)$ shows summability by the 3rd Cesàro mean [1. p. 7],

$$W^{*2}(t) = (\sum_\lambda w_\lambda^2)^{-1}\int_{-\infty}^{\infty} W(t-s)W(s)ds,$$

$$\dot{W}(T_\mu) = \frac{d}{dt}W(t)\Big|_{t=T_\mu},$$

$$\ddot{W}(T_\mu) = \frac{d^2}{dt^2}W(t)\Big|_{t=T_\mu},$$

$$\dot{W}^{*2}(T_\mu) = \frac{d}{dt}W^{*2}(t)\Big|_{t=T_\mu},$$

$$\ddot{W}^{*2}(T_\mu) = \frac{d^2}{dt^2}W^{*2}(t)\Big|_{t=T_\mu},$$

* This local approximation of the frequency response function constitutes the basis of our present analysis. Its extension to the cases with the polynomial factor of higher degrees is straightforward.

and the existence of these quantities is assumed. The second term of this last expression of $E(\hat{A}(f_\mu))$ represents the contribution of $E(x/(x+y)) - E(x)/E(x+y)$ and usually is very small while the first term represents the main effect of the smoothing operation. Hereafter, we restrict our attension to this first term. This term is analogous to that obtained in the preceding paper [1, p. 7] by the analysis of the bias in the estimation of the power spectral density. However, there is one important difference in that $W(0)$, $\dot{W}(0)$ and $\ddot{W}(0)$ of the formers are replaced by $W(T_\mu)$, $\dot{W}(T_\mu)$ and $\ddot{W}(T_\mu)$ in the present formulae, respectively.

Taking into account that $2\pi T_\mu$ represents $-d[\arg A(f)]/df|_{f=f_\mu}$, we can see that this appearance of T_μ in the formula is showing that the allignment of the phase shift between the output and input is essential for the estimation of $A(f)$. This fact was entirely disregarded in the paper by Goodman [8]. As to the lag windows of trigonometric sum type hitherto used, $W(T_\mu)$ is less than one except for the case where $T_\mu=0$ holds, hence even if the contributions of the term βf_ν, βf_ν^2 are small the estimate may suffer a significant decrease in the amplitude gain by the contribution of T_μ.

Taking into account the very rapid change of phase shift of the frequency response function of the ship model treated by Darzell and Yamonouchi [6], this gives a theoretical explanation to the apparently very low amplitude gain, accordingly to the apparently very low coherency, observed there at the frequencies where the amplitude gain showed a significant peak in the analysis of the response of the ship model to the waves.

The present representation of the bias suggests a method how to remedy this defect of the window. For we can see that if we replace w_ν by $w_\nu \exp(-2\pi i f_\nu T_\mu)$ or $W(t)$ by $W(t-T_\mu)$ then we have

$$\sum_\nu w_\nu \exp(-2\pi i f_\nu T_\mu) A(f_\mu - f_\nu) = \alpha\left(W(0) + \frac{\beta}{2\pi i}\dot{W}(0) + \frac{\gamma}{(2\pi i)^2}\ddot{W}(0)\right) \; (C,\ 3),$$

and if we use the window $W_k(2, *)$ which was defined in the preceding paper and for which $\dot{W}(0) = \ddot{W}(0) = 0$ holds, we have

$$\sum_\nu w_\nu \exp(-2\pi i f_\nu T_\mu) A(f_\mu - f_\nu) = \alpha \; (C.\ 3).$$

However, if we try to use this procedure practically, we have to put an estimate of T_μ in place of T_μ itself, and this suggests the necessity of the use of lag windows with somewhat flat tops. It is known that in a physically realizable minium-phase system there exists functional relationship between the amplitude gain and the phase shift, so to know of T_μ is actually to know of the shape of the amplitude gain near the frequency f_μ, which we are struggling to obtain. Fortunately, however, it can be seen, by inserting some concrete values into the present evaluation formula of the bias, that even when $|T_\mu|$ is fairly large $W(T_\mu) -(\gamma/(2\pi))^2\ddot{W}(T_\mu)$ may sometimes be much greater then $|(\beta/2\pi)\dot{W}(T_\mu)|$ and the bias of the phase shift due to the smoothing is relatively small. This fact and the following consideration of the overall reduction in the gain shows that we can usually get a reliable estimate of the phase shift and thus of T_μ, and the estimate is used to obtain the final estimate of $A(f_\mu)$.

To get an overall view of the effect of the smoothing on the frequency response function we make an analysis in the time domain [3]. It is assumed that $A(f)$ is the Fourier transform of some function $h(t)$ and is represented as

$$A(f)=\int_{-T}^{T}\exp(-2\pi ift)h(t)dt\,.$$

Then we have

$$\sum_\nu w_\nu A(f_\mu-f_\nu)=\int_{-T}^{T}\exp(-2\pi if_\mu t)W(t)h(t)dt\,.$$

and

$$\left(\frac{1}{2T}\right)\sum_\mu\Big|\sum_\nu w_\nu A(f_\mu-f_\nu)\Big|^2=\int_{-T}^{T}|W(t)h(t)|^2dt$$

$$\left(\frac{1}{2T}\right)\sum_\mu|A(f_\mu)|^2=\int_{-T}^{T}|h(t)|^2dt\,.$$

From these formulae and the fact that $W(t)$ is usually less than unity, it can be seen that, to keep minimum the overall reduction in the gain, $W(t)$ which gives the maximum of $\int_{-T}^{T}|W(t)h(t)|^2dt$ should be used. In the practical application of our estimation procedure we can observe the sample crosscovariance function which is a convolution of $h(t)$ with the sample autocovariance function of $x(t)$ distorted by the existence of some additive disturbance $n(t)$, and the sample crosscovariance function provides some information of the shape of $h(t)$, especially when some type of overall prewhitening operation is applied to the input.* Therefore in case where the input is considered to be fairly white in the range of our concern, we can obtain a fairly good overall estimate of $A(f)$ by shifting the center of the lag window or the origin of the time axis of the sample crosscovariance function to that time point t where the maximum of the absolute value of the sample crosscovariance function occurs. From the result of the foregoing analysis, it can be seen that

* As to the overall prewhitening operation, some numerical examples are illustrated in the former paper by one of the authors [2].

the local whiteness of the input is desired to keep the estimate of the frequency response function unbiased and overall whiteness is only necessary to get a good primary estimate.

We shall here analyse the effect of the smoothing operation for a certain concrete type of $A(f)$. We assume

$$h(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}\exp(-\zeta\omega_n t)\sin\sqrt{1-\zeta^2}\,\omega_n t) \qquad t\geqq 0$$

$$=0 \qquad t<0.$$

Then we have

$$A(f)=\frac{\omega_n^2}{(\omega_n^2-(2\pi f)^2)+2\zeta\omega_n i(2\pi f)}=|A(f)|\exp(i\phi)$$

$$\phi=\tan^{-1}\left(\frac{-2\zeta\omega_n 2\pi f}{\omega_n^2-(2\pi f)^2}\right).$$

This $A(f)$ is the frequency response function of the system of which behavior is described by the second order differential equation

$$\left(\tilde{T}\frac{d^2}{dt^2}+\frac{d}{dt}+K\right)\theta_O(t)=K\theta_I(t)$$

where $\theta_I(t)$ and $\theta_O(t)$ represent the input and output of the system and

$$\omega_n^2=\frac{K}{\tilde{T}}$$

$$\zeta=\frac{2}{2\sqrt{K\tilde{T}}}.$$

Consider the case where $\zeta\ll 1$ or very lightly damped oscillating system with one degree of freedom. In this case $|A(f)|$ shows its peak near its natural resonant frequency $f_n=\omega_n/2\pi$, and a value of T_μ at $f_\mu=f_n$, which appeared in the local approximation formula of $A(f)$, will be given by

$$T_\mu=-\frac{1}{2\pi}\frac{d\phi}{df}\bigg|_{f=f_n}=\frac{1}{\zeta\omega_n}.$$

Further, if we define $\Delta f_+(>0)$ and $\Delta f_-(>0)$ by

$$|A(f_n+\Delta f_+)|=|A(f_n-\Delta f_-)|=\frac{1}{\sqrt{2}}|A(f_n)|,$$

then under the condition $\zeta\ll 1$ we have

$$\Delta f_+ + \Delta f_- \doteqdot \frac{\zeta\omega_n}{\pi} .$$

From these two results we get the relation

$$T_\mu \doteqdot \frac{1}{\pi} \frac{1}{\Delta f_+ + \Delta f_-} .$$

For example, when $\Delta f_+ + \Delta f_- = 1$ c.p.s. we have to shift the window $1/\pi$ second to the right of origin. This $\Delta f_+ + \Delta f_-$ is usually called the bandwidth of the filter or of $A(f)$ and our present result shows that the amount of shift T_μ is inversely proportional to the bandwidth of the filter. This knowledge will prevent the danger of using a lag window with $W(T_\mu) \doteqdot 0$. Now we evaluate the bias induced on the present $A(f)$ by the use of windows of trigonometric sum type. It is assumed that every window is shifted by the amount $K = 1/\zeta\omega_n$ and that $T_m > K$. Adopt the approximation

$$A(f_n + f) \doteqdot \frac{-i\omega_n}{2} \frac{1}{\zeta\omega_n + i2\pi f}$$

around the natural frequency f_n. Then we have

$$\frac{1}{2T} \sum_\nu A(f_n - f_\nu) w_\nu \doteqdot \frac{-i\omega_n}{2} \int_0^\infty \exp(-\zeta\omega_n t) W_K(t) dt$$

$$= A(f_n)(\zeta\omega_n) \int_0^\infty \exp(-\zeta\omega_n t) W_K(t) dt .$$

Now

$$\int_0^\infty \exp(-\zeta\omega_n t) W_K(t) \zeta\omega_n dt$$

$$= \sum_{n=-k}^{k} a_n \int_0^{K+T_m} \exp\left(-2\pi i \frac{\nu}{2T_m}(t-K)\right) \exp(-\zeta\omega_n t)\, \zeta\omega_n dt$$

and when we define

$$\rho = \frac{\pi}{\zeta\omega_n} \frac{1}{T_m} \left(\doteqdot \frac{1}{\Delta f_+ + \Delta f_-} \cdot \frac{1}{T_m} \right)$$

we get for $k=3$

$$A_D(\zeta) = \int_0^\infty \exp(-\zeta\omega_n t) W_K(t) \zeta\omega_n dt$$

$$= a_0 \left\{ 1 - \exp\left(-\left(1 + \frac{\pi}{\rho}\right)\right)\right\}$$

TABLE 1. Comparison of biases induced on $A(f)$

	$W_1(0, \alpha)$	$W_1(0, \beta/\alpha)$	$W_1(0, \alpha\beta)$	$W_2(0, \alpha)$
a_0	0.3333	0.5272	0.5132	0.2000
$a_1=a_{-1}$	0.3333	0.2364	0.2434	0.2000
$a_2=a_{-2}$	*	*	*	0.2000
$a_3=a_{-3}$	*	*	*	*
$A_D(2)$	0.5050	0.6267	0.6179	0.2146
$A_D(1)$	0.7938	0.8492	0.8452	0.5873
$A_D(1/2)$	0.9293	0.9497	0.9482	0.8338
$A_D(1/4)$	0.9800	0.9859	0.9855	0.9456

	$W_2(2, \alpha)$	$W_2(2, \beta/\alpha)$	$W_2(2, \alpha\beta)$	$W_3(2, \alpha)$
a_0	0.4857	0.4675	0.6398	0.3333
$a_1=a_{-1}$	0.3429	0.2350	0.2401	0.2857
$a_2=a_{-2}$	−0.0587	−0.0588	−0.0600	0.1429
$a_3=a_{-3}$	*	*	*	−0.0952
$A_D(2)$	0.6893	0.7630	0.7594	0.4169
$A_D(1)$	0.9097	0.9330	0.9319	0.8171
$A_D(1/2)$	0.9804	0.9862	0.9860	0.9496
$A_D(1/4)$	0.9978	0.9983	0.9984	0.9916

$$+a_1\frac{2}{1+\rho^2}\left\{\cos\rho+\rho\sin\rho+\exp\left(-\left(1+\frac{\pi}{\rho}\right)\right)\right\}$$

$$+a_2\frac{2}{1+(2\rho)^2}\left\{\cos 2\rho+2\rho\sin 2\rho-\exp\left(-\left(1+\frac{\pi}{\rho}\right)\right)\right\}$$

$$+a_3\frac{2}{1+(3\rho)^2}\left\{\cos 3\rho+3\rho\sin 3\rho+\exp\left(-\left(1+\frac{\pi}{\rho}\right)\right)\right\}.$$

The result of numerical computation of this $A_D(\zeta)$ for the windows treated in the preceding paper [1] is given in table 1, for the values of $\rho=2$, 1, 1/2 and 1/4 which correspond to the values of $T_m=\pi\tilde{T}$, $2\pi\tilde{T}$, $4\pi\tilde{T}$ and $8\pi\tilde{T}$. The result shows that ρ must be kept less than 1 for the ordinary purpose of estimation and that the window $Q(a_0=0.64$ $a_1=a_{-1}=0.24$ $a_2=a_{-2}=-0.06)$ introduced in the preceding paper, which is approximately the same with $W_2(2, \alpha\beta)$, will be quite suited for ordinary use.

4. Sampling variability of the estimate of coherency

As was seen in section 2, the value of coherency is necessary for eʌaluation of the accuracy of our estimate of the frequency response by the use of windows of trigonometric sum type.

$W_2(0, \beta/\alpha)$	$W_2(0, \alpha\beta)$	$W_3(0, \alpha)$	$W_3(0, \beta/\alpha)$	$W_3(0, \alpha\beta)$
0.6202	0.4282	0.1429	0.6129	0.4229
0.2358	0.2433	0.1429	0.2182	0.2124
−0.0459	0.0426	0.1429	−0.0431	0.0434
*	*	0.1429	0.0184	0.0327
0.7326	0.5206	0.1498	0.7135	0.4959
0.9145	0.7851	0.4039	0.8822	0.7335
0.9782	0.9219	0.7336	0.9610	0.8942
0.9957	0.9762	0.9030	0.9885	0.9646

$W_3(2, \beta/\alpha)$	$W_3(2, \alpha\beta)$	$W_3(4, \alpha)$	$W_3(4, \beta/\alpha)$	$W_3(4, \alpha\beta)$
0.6643	0.5571	0.5671	0.7085	0.7029
0.2306	0.2610	0.3247	0.2186	0.2228
−0.0609	−0.0190	−0.1299	−0.0875	−0.0891
0.0041	−0.0205	0.0216	0.0146	0.0149
0.7793	0.6780	0.7725	0.8218	0.8198
0.9385	0.9048	0.9375	0.9526	0.8198
0.9879	0.9777	0.9890	0.9923	0.9923
0.9987	0.9969	0.9992	0.9994	0.9996

function. In practical applications, it is often more necessary for making a decision whether such analysis of linear relation between $x(t)$ and $y(t)$ has a practical meaning or not. We define sample coherency $\hat{\gamma}^2(f)$ at f_μ by

$$\hat{\gamma}^2(f_\mu)=|\hat{A}(f_\mu)|^2\frac{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}{\sum_\nu w_\nu|Y(f_\mu-f_\nu)|^2}.$$

This $\hat{\gamma}^2(f_\mu)$ will be an estimate of the true coherency $\gamma^2(f_\mu)$, and the sampling variability of this estimate will be treated in the following. It was shown in the preceding section that the main trouble in the estimation is the bias of the estimate, and that by using the shift operation and lag window, properly combined, we can evade this difficulty. In this section we discuss the sampling variability of the estimate of the coherency under the extremely simplified condition of constant $A(f)$ and constant $p_x(f)$ and $w_\nu=1/n(\nu=1, 2, 3, \cdots, n)$. This condition is identical to that adopted in the paper [8] by Goodman, and we can show that in this case our estimation procedure reduces to that of the classical regression estimate and all the necessary information is directly available from the standard text of mathematical statistics. We adopt the notations

$$U_{X,\nu}=R_e X\left(\frac{\nu}{2T}\right)$$

$$V_{X,\nu}=I_m X\left(\frac{\nu}{2T}\right)$$

$$U_{Y,\nu}=R_e Y\left(\frac{\nu}{2T}\right)$$

$$V_{Y,\nu}=I_m Y\left(\frac{\nu}{2T}\right)$$

$$U_{N,\nu}=R_e N\left(\frac{\nu}{2T}\right)$$

$$V_{N,\nu}=I_m N\left(\frac{\nu}{2T}\right)$$

$$A=R_e A(f)$$

$$B=I_m A(f)$$

$$\sigma_N^2=\frac{1}{2}\,p_n(f)$$

$$\sigma_X^2=\frac{1}{2}\,p_x(f)$$

$$\sigma_Y^2=\frac{1}{2}\,p_y(f)=|A(f)|^2\sigma_X^2+\sigma_N^2$$

$$\gamma^2=\text{coherency}=\frac{|A(f)|^2\sigma_X^2}{\sigma_Y^2}=(A^2+B^2)\left(\frac{\sigma_X}{\sigma_Y}\right)^2$$

where R_e and I_m denote "the real part of" and "the imaginary part of", respectively. Following the analysis in section 2 we have

$$U_{Y,\nu}=AU_{X,\nu}-BV_{X,\nu}+U_{N,\nu}$$
$$V_{Y,\nu}=BU_{X,\nu}+AV_{X,\nu}+V_{N,\nu}$$

and it is assumed that $U_{X,\nu}$, $V_{X,\nu}$, $U_{N,\nu}$, $V_{N,\nu}$ are mutually independent Gaussion random variables with zero means and variances $D^2(U_{X,\nu})=D^2(V_{X,\nu})=\sigma_X^2$ $D^2(U_{N,\nu})=D^2(V_{N,\nu})=\sigma_N^2=\sigma_Y^2(1-\gamma^2)$. The 4-dimensional probability density function $\phi(u_y, v_y, u_x, v_x)$ of $(U_{Y,\nu}, V_{Y,\nu}, U_{X,\nu}, V_{X,\nu})$ at $(u_y, v\ , u_x, v_x)$ is, therefore, given by

$$\phi(u_y, v_y, u_x, v_x)=\frac{1}{2\pi\sigma_X^2}\exp\left(-\frac{1}{2\sigma_X^2}\{u_x^2+v_x^2\}\right)$$
$$\times\frac{1}{2\pi\sigma_Y^2(1-\gamma^2)}\exp\left(-\frac{1}{2\sigma_Y^2(1-\gamma^2)}\{(u_y-Au_x+Bv_x)^2+(v_y-Bu_x-Av_x)^2\}\right).$$

Now, for a random sample $((U_{Y,\nu}, V_{Y,\nu}, U_{X,\nu}, V_{X,\nu});\ \nu=\nu_0+1, \nu_0+2, \cdots, \nu_0+n)$ the maximum-likelihood estimates $\hat{A}$ and $\hat{B}$ of A and B are given by

$$\hat{A}=\frac{\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{Y,\nu}U_{X,\nu}+\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{Y,\nu}V_{X,\nu}}{\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{X,\nu}^2+\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{X,\nu}^2}, \qquad \hat{B}=\frac{\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{Y,\nu}U_{X,\nu}-\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{Y,\nu}V_{X,\nu}}{\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{X,\nu}^2+\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{X,\nu}^2}$$

and the sample coherency $\hat{\gamma}^2$ is given as

$$\hat{\gamma}^2=\frac{(\hat{A}^2+\hat{B}^2)(\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{X,\nu}^2+\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{X,\nu}^2)}{\sum_{\nu=\nu_0+1}^{\nu_0+n} U_{Y,\nu}^2+\sum_{\nu=\nu_0+1}^{\nu_0+n} V_{Y,\nu}^2}.$$

After a moment's consideration, we come to know that we are making here a regression analysis of the form

$$Y=AX_1+BX_2+\varepsilon$$

where

$$Y=(U_{Y,\nu_0+1}, U_{Y,\nu_0+2}, \cdots, U_{Y,\nu_0+n}, V_{Y,\nu_0+1}, V_{Y,\nu_0+2}, \cdots, V_{Y,\nu_0+n})'$$
$$X_1=(U_{X,\nu_0+1}, U_{X,\nu_0+2}, \cdots, U_{X,\nu_0+n}, V_{X,\nu_0+1}, V_{X,\nu_0+2}, \cdots, V_{X,\nu_0+n})'$$
$$X_2=(-V_{X,\nu_0+1}, -V_{X,\nu_0+2}, \cdots, -V_{X,\nu_0+n}, U_{X,\nu_0+1}, U_{X,\nu_0+2}, \cdots, U_{X,\nu_0+n})'$$
$$\varepsilon=(U_{N,\nu_0+1}, U_{N,\nu_0+2}, \cdots, U_{N,\nu_0+n}, V_{N,\nu_0+1}, V_{N,\nu_0+2}, \cdots, V_{N,\nu_0+n})'$$

where $'$ denotes the transpose of the vector. Statistical properties of $\hat{A}$ and $\hat{B}$ are well known and the sample coherency $\hat{\gamma}^2$ is the square of multiple correlation coefficient between (X_1, X_2) and Y. Taking into account of the fact that the inner product $\langle X_1, X_2\rangle$ of vectors X_1 and X_2 is identically equal to zero, we can at once get the representation

$$\langle\varepsilon, \varepsilon\rangle=(\hat{A}-A)^2\langle X_1, X_1\rangle+(\hat{B}-B)^2\langle X_2, X_2\rangle+\langle\hat{\varepsilon}, \hat{\varepsilon}\rangle$$

where $\hat{\varepsilon}=Y-\hat{A}X_1-\hat{B}X_2$ and $\langle,\rangle$ denotes the inner product of vectors. Fisher's lemma can be applied [5, p. 379] to this representation and we know that the right hand side members, when divided by σ_N^2, are mutually independently distributed as χ^2's with d.f. 1, 1 and $2(n-1)$, respectively. The sample coherency $\hat{\gamma}^2$ can be represented as

$$\hat{\gamma}^2=\frac{(\hat{A}^2\langle X_1, X_1\rangle+B\langle X_2, X_2\rangle)(\sigma_N^2)^{-1}}{(\hat{A}^2\langle X_1, X_1\rangle+\hat{B}^2\langle X_2, X_2\rangle+\langle\hat{\varepsilon}, \hat{\varepsilon}\rangle)(\sigma_N^2)^{-1}}$$

and we can see that the numerator is a non-central χ^2-variable with d.f.

2 and non-centrality $\gamma^2\langle X_1, X_2\rangle/(1-\gamma^2)\sigma_X^2$. Taking into account the fact that $\langle X_1, X_1\rangle/\sigma_X^2$ is distributed as a χ^2-variable with d.f. $2n$, we can see that the distribution of the sample coherency $\hat{\gamma}^2$ is that of the square of the sample multiple correlation coefficient when the corresponding population value is equal to γ^2 and a straightforward calculation shows that its probability density $p(\zeta)$ at $\hat{\gamma}^2=\zeta$ is given by

$$p(\zeta)=(1-\eta)^n\sum_{k=0}^{\infty}\frac{1}{\Gamma(n)\Gamma(n-1)}\left(\frac{\Gamma(n+k)}{\Gamma(k+1)}\right)^2\eta^k\zeta^{k+1-1}(1-\zeta)^{n-1-1}$$

where $\eta=\gamma^2$. This is identical with the formula (4.60) of Goodman's paper [8] if z^2 in the latter is replaced by ζ*.

This $p(\zeta)$ will give an approximation to the true distribution of the sample coherency $\hat{\gamma}^2(f_\mu)$ for general spectral window $\{w_\nu\}$, and for the analysis of the sampling variability of the sample coherency $\hat{\gamma}^2(f_\mu)$, obtained by using a lag window of trigonometric sum type, one should take the nearest integer to $(2T/T_m)(1/2\sum_{n=k}^{k}|a_n^2|)$ as the n in the above consideration. For the sake of typographical simplicity we hereafter use the letter ζ in place of $\hat{\gamma}^2$. Now, $E(\zeta)$ and $E(\zeta^2)$ are given by

$$E(\zeta)=(1-\eta)^n\sum_{k=0}^{\infty}\frac{\Gamma(n+k)}{\Gamma(n)\Gamma(k+1)}\,\frac{k+1}{k+n}\,\eta^k$$

$$E(\zeta^2)=(1-\eta)^n\sum_{k=0}^{\infty}\frac{\Gamma(n+k)}{\Gamma(n)\Gamma(k+1)}\,\frac{k+1}{k+n}\,\frac{k+2}{k+n+1}\,\eta^k\,.$$

Numerical computations of $E(\zeta)$ and $D^2(\zeta)$ by these formulae for values of n and η which will often be met in practical applications are tabulated in table 2.

Fig. 1 shows the results of a sampling experiment of estimation of the coherency using the windows hamming and Q. Necessary information about the true $A(f)$ of this case will be given in fig's in section 6. It seems that, at least for this example, $E(\zeta)$ and $D^2(\zeta)$ obtained by using the present distribution of ζ provide us with good estimates of the corresponding true values of the distribution of the sample coherency. However, it must be borne in mind that this estimation is valid only when the bias due to smoothing is negligibly small.

For $\rho=\dfrac{\zeta}{1-\zeta}$ we have

$$\rho_0\equiv E(\rho)=\frac{n}{n-2}\,\bar{\rho}+\frac{1}{n-1}$$

* In Goodman's paper, Z and z in (4.8.6.) and (4.16.6) should be read as Z^2 and z^2 respectively.

$$D^2(\rho) = \frac{2n(n-1)}{(n-2)^2(n-3)}\left(\bar{\rho}^2+\bar{\rho}+\frac{1}{2n}\right)$$

where

$$\bar{\rho}=\frac{\eta}{1-\eta};$$

The factor $\bar{\rho}^{-1}=(1-\eta)/\eta$ is necessary for evaluation of the relative accuracy of the estimate of the frequency response function and sometimes we have to substitute its sample value $\rho^{-1}=(1-\zeta)/\zeta$ for this. For evaluation of the error of this last value, the knowledge of $E(\zeta)$ and $D^2(\zeta)$ will conveniently be used. From the relation $\zeta=\rho/(1+\rho)$ we have for $\rho=\rho_0+\Delta\rho$ the local approximation

$$\zeta \doteqdot \frac{\rho_0}{1+\rho_0}+\left(\frac{d\zeta}{d\rho}\right)_{\rho=\rho_0}\Delta\rho.$$

Using this approximation we have

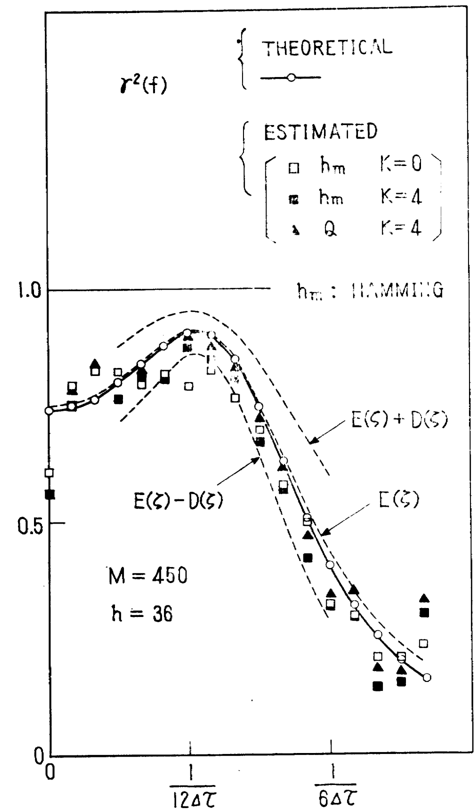


Fig. 1. Sampling experiment of estimation of coherency.

$$E(\zeta)=\frac{\rho_0}{1+\rho_0}$$

$$D^2(\zeta)\doteqdot\frac{2n(n-1)}{(n-2)^2(n-3)}\,\frac{(\bar{\rho}^2+\bar{\rho}+1/2n)}{(1+\zeta_0)^4}\,.$$

In table 2 the values obtained by using the present approximation are compared with the corresponding theoretical values. The result is also illustrated in fig. 2. It can be seen that, in this range of parameters, our formulae provide us with estimates which are accurate enough for the purpose of practical applications. Moreover, our approximation shows that $D^2(\zeta)$ is approximately inversely proportional to n, this is quite natural and useful for the design of experiment and analysis. The above result of sampling experiment seems to show that the simplified condition of this section gives a fairly reasonable approximation to the true distribution of the estimate of coherency. From this it can be inferred that it will not be quite useless to mention here of the confidence region

TABLE 2. Comparison of the approximate values with their corresponding theoretical values.

η	$E(\zeta)$					
	$n=5$		$n=10$		$n=20$	
	A*)	T**)	A	T	A	T
0.4	0.59	0.48	0.49	0.44	0.44	0.42
0.5	0.67	0.56	0.58	0.53	0.54	0.51
0.6	0.74	0.64	0.67	0.62	0.63	0.61
0.7	0.81	0.72	0.75	0.71	0.73	0.70
0.8	0.88	0.81	0.84	0.80	0.82	0.80
0.9	0.94	0.90	0.92	0.90	0.91	0.90

η	$D(\zeta)$					
	$n=5$		$n=10$		$n=20$	
	A	T	A	T	A	T
0.4	0.27	0.21	0.17	0.16	0.12	0.12
0.5	0.24	0.20	0.16	0.15	0.11	0.11
0.6	0.20	0.18	0.14	0.14	0.10	0.10
0.7	0.15	0.16	0.11	0.11	0.08	0.08
0.8	0.10	0.12	0.08	0.08	0.06	0.06
0.9	0.05	0.07	0.04	0.05	0.03	0.03

η	$C.V.(\zeta)=E(\zeta)^{-1}D(\zeta)$					
	$n=5$		$n=10$		$n=20$	
	A	T	A	T	A	T
0.4	0.46	0.44	0.36	0.36	0.28	0.28
0.5	0.36	0.36	0.28	0.29	0.21	0.21
0.6	0.27	0.29	0.21	0.22	0.15	0.16
0.7	0.19	0.22	0.14	0.16	0.11	0.11
0.8	0.12	0.15	0.09	0.10	0.07	0.07
0.9	0.06	0.08	0.04	0.05	0.03	0.03

*) A: approximate
**) T: theoretical

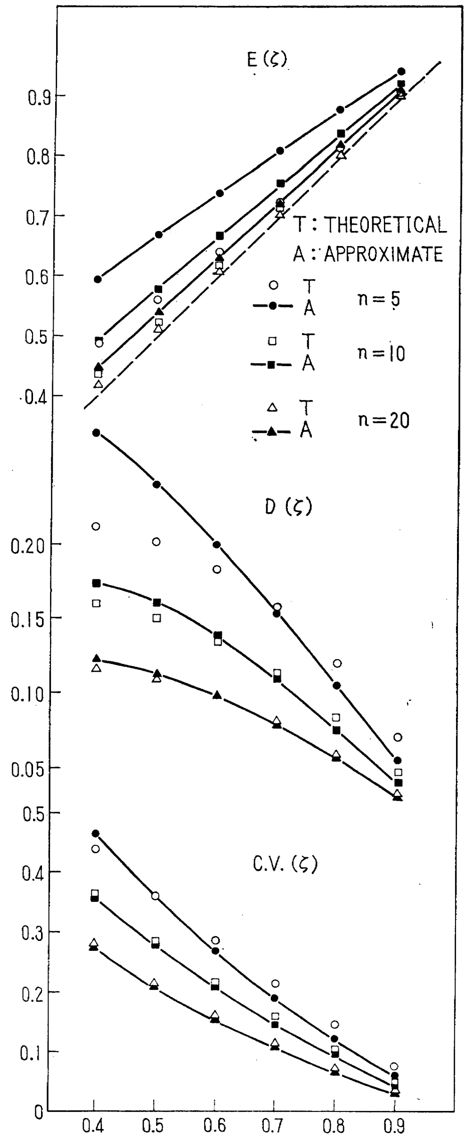


Fig. 2. Comparison of the approximate values with their corresponding theoretical values.

of the estimate of the frequency response function under that condition. We have obtained the relation

$$\langle \varepsilon, \varepsilon \rangle = (\hat{A}-A)^2 \langle X_1, X_1 \rangle + (\hat{B}-B) \langle X_2, X_2 \rangle + \langle \hat{\varepsilon}, \hat{\varepsilon} \rangle$$

where the right-hand side members, when divided by σ_N^2, are mutually independently distributed according to the χ^2-distributions with d.f. 1, 1 and $2(n-1)$, respectively. From this relation we can see that the statistic F which is defined by

$$F=(n-1)\frac{\{(\hat{A}-A)^2+(\hat{B}-B)^2\}\langle X, X\rangle}{\langle\hat{\varepsilon}, \hat{\varepsilon}\rangle},$$

where $\langle X, X\rangle=\langle X_1, X_1\rangle=\langle X_2, X_2\rangle$, is distributed according to the F-distribution with d.f.s 2 and $2(n-1)$.

Therefore the confidence region S of (A, B) with confidence coefficient δ is given as follows.

$$S=\left\{(\alpha, \beta);\ (\hat{A}-\alpha)^2+(\hat{B}-\beta)^2\leqq\frac{\langle\hat{\varepsilon}, \hat{\varepsilon}\rangle}{(n-1)\langle X, X\rangle}F(\delta, 2, 2(n-1))\right\}$$

where $F(\delta, 2, 2(n-1))$ is defined by the relation

$$\text{Prob}\ (F\leqq F(\delta, 2, 2(n-1))=\delta.$$

When the lag window of trigonometric sum type is used to get $\hat{A}(f_\mu)$, $\langle\hat{\varepsilon}, \hat{\varepsilon}\rangle/\langle X, X\rangle$ and n in this expression are replaced by

$$(1-\hat{\gamma}^2(f_\mu))\frac{\sum_\nu w_\nu|Y(f_\mu-f_\nu)|^2}{\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2}$$

and the nearest integer to $(2T/T_m)(1/2\sum_{n=-k}^{k}a_n^2)$, respectively, to obtain an approximate region R to the above confidence region for $A(f_\mu)$ with confidence coefficient δ, i.e., the region R in the complex domain is obtained from the following relation.

$$R=\left\{G;\ |\hat{A}(f_\mu)-G|^2\leqq\frac{1}{n-1}\left(\frac{\hat{p}_{yy}(f_\mu)}{\hat{p}_{xx}(f_\mu)}-|\hat{A}(f_\mu)|^2\right)F(\delta, 2, 2(n-1))\right\}$$

where

$$n=\text{the nearest integer to } \frac{2T}{T_m}\frac{1}{2\sum_{n=-k}^{k}a_n^2}$$

$$\hat{p}_{yy}(f_\mu)=\sum_\nu w_\nu|Y(f_\mu-f_\nu)|^2$$

$$\hat{p}_{xx}(f_\mu)=\sum_\nu w_\nu|X(f_\mu-f_\nu)|^2.$$

We will here briefly mention about practical meaning of the use of the present confidence region. The results of analysis of sampling variabilities of $\hat{A}(f_\mu)$ and $\hat{\gamma}^2(f_\mu)$ in this and former sections are used mainly for the design of experiment and are taken into consideration *before* the data are obtained. On the other hand *after* the data have been obtained the confidence region will indicate the precision of our estimate $\hat{A}(f_\mu)$ by using the information contained in the present data, where $X(f_\mu)$'s are treated as fixed variables. It can be seen, as was stated in the former paper [2, p. 142], that in these circumstances we may apply, without disturbing the sampling variability of the estimate $\hat{A}(f_\mu)$, any sort of prewhitening operation to the data of input $\{x(t)\}$ after they are obtained, if only the operation is not dependent on the records of output $\{y(t)\}$.

The setup adopted in this section is a quite rough approximation to the real state where the lag window of trigonometric sum type is used and its utility must be checked by some artificial sampling experiments.

5. Computation scheme for the estimation of the frequency response function

In this section a computation scheme will be described to make our estimation procedure of the frequency response function practical. First we shall slightly modify the form of our estimate. Our estimate $\hat{A}(f_\mu)$ of the frequency response function was defined as

$$\hat{A}(f_\mu)=\frac{\sum_\nu w_\nu Y(f_\mu-f_\nu)\overline{X(f_\mu-f_\nu)}}{\sum_\nu w_\nu |X(f_\mu-f_\nu)|^2}.$$

We have

$$\begin{aligned}&\sum_\nu w_\nu Y(f_\mu-f_\nu)\overline{X(f_\mu-f_\nu)}\\ &=\sum_\nu w_\nu \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\mu-\nu}{2T}t\right)y(t)\exp\left(2\pi i\frac{\mu-\nu}{2T}s\right)x(s)dtds\\ &=\sum_\nu w_\nu \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\mu-\nu}{2T}(t-s)\right)y(t)x(s)dtds\\ &=\sum_\nu w_\nu \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\mu-\nu}{2T}\tau\right)\tilde{y}(\tau+s)x(s)dsd\tau\end{aligned}$$

where

$$\begin{aligned}\tilde{y}(t)&=y(t-2T)\quad\text{when}\quad t>T\\ &=y(t+2T)\quad\text{when}\quad t<-T.\end{aligned}$$

Therefore if we put

$$\tilde{C}_{yx}(\tau)=\frac{1}{2T}\int_{-T}^{T}\tilde{y}(\tau+s)x(s)ds$$

we have

$$\sum_\nu w_\nu Y(f_\mu - f_\nu)\overline{X(f_\mu - f_\nu)} = \sum_\nu w_\nu \int_{-T}^{T} \exp\left(-2\pi i \frac{\mu-\nu}{2T}\tau\right)\tilde{C}_{yx}(\tau)d\tau$$

$$= \int_{-T}^{T} \exp\left(-2\pi i \frac{\mu}{2T}\tau\right) W(\tau)\tilde{C}_{yx}(\tau)d\tau$$

$$= \int_{-T_m}^{T_m} \exp\left(-2\pi i \frac{\mu}{2T}\tau\right) W(\tau)\tilde{C}_{yx}(\tau)d\tau\,.$$

We define here $C_{yx}(\tau)$ by

$$C_{yx}(\tau) = \frac{1}{2T}\int_{-T}^{T-\tau} y(\tau+s)x(s)ds \qquad \tau \geqq 0$$

$$= \frac{1}{2T}\int_{-T-\tau}^{T} y(\tau+s)x(s)ds \qquad \tau < 0$$

and $\bar{C}_{yx}(\tau)$ by

$$\bar{C}_{yx}(\tau) = \tilde{C}_{yx}(\tau) - C_{yx}(\tau).$$

$$\bar{C}_{yx}(\tau) = \frac{1}{2T}\int_{T-\tau}^{T} y(\tau+s-2T)x(s)ds \qquad \tau \geqq 0$$

$$= \frac{1}{2T}\int_{-T}^{-T-\tau} y(\tau+s+2T)x(s)ds \qquad \tau < 0\,.$$

If in the above obtained expression of $\sum_\nu w_\nu Y(f_\mu - f_\nu)\overline{X(f_\mu - f_\nu)}$ we replace $\hat{C}_{yx}(\tau)$ by $C_{xy}(\tau)$, and if T is sufficiently large so that $Ey(s)x(s)$, $Ey(t)y(s)$, and $Ex(t)x(s)$ can all be considered to be negligibly small for $|t-s| > 2(T-T_m)$, we obtain a statistic which has almost the same mean as that of the former one and has a variance smaller than that of the former, i.e., the contribution of $\bar{C}_{yx}(\tau)$ to $\sum_\nu w_\nu Y(f_\mu - f_\nu)\overline{X(f_\mu - f_\nu)}$ can be considered to have a zero-mean and to be orthogonal to the contribution of $C_{yx}(\tau)$. If we correspondingly replace $\sum_\nu w_\nu |X(f_\mu - f_\nu)|^2$ in the definition of $\hat{A}(f_\mu)$ by

$$\int_{-T}^{T} \exp\left(-2\pi i \frac{\mu}{2T}\tau\right) W(\tau)C_{xx}(\tau)d\tau$$

where $C_{xx}(\tau)$ is by definition

$$C_{xx}(\tau) = \frac{1}{2T}\int_{-T}^{T-|\tau|} x(|\tau|+t)x(t)dt,$$

then our new estimate $\hat{\hat{A}}(f_\mu)$ is given by

$$\hat{\hat{A}}(f_\mu)$$
$$=\left(\int_{-T}^{T}\exp\left(-2\pi i\frac{\mu}{2T}\tau\right)W(\tau)C_{yx}(\tau)d\tau\right)\left(\int_{-T}^{T}\exp\left(-2\pi i\frac{\mu}{2T}\tau\right)W(\tau)C_{xx}(\tau)d\tau\right)^{-1}.$$

This $\hat{\hat{A}}(f_\mu)$ can be represented in the form

$$\hat{\hat{A}}(f_\mu)=\frac{\sum_\nu \tilde{w}_\nu Y(\mu/2T-\nu/4T)\overline{X(\mu/2T-\nu/4T)}}{\sum_\nu \tilde{w}_\nu |X(\mu/2T-\nu/4T)|^2}$$

where

$$\tilde{w}_\nu=\frac{1}{4T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{4T}t\right)W(t)dt$$

$$X(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(-2\pi ift)x(t)dt$$

$$Y(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(-2\pi ift)y(t)dt\,.$$

When $A(f)$ can be considered to be nearly a constant in the effective range of the smoothing operation, we can get from this expression of $\hat{\hat{A}}(f_\mu)$ the relation which has formed the basis of the analysis of this paper :

$$\hat{\hat{A}}(f_\mu)\approx A(f_\mu)+\frac{\sum_\nu \tilde{w}_\nu N(\mu/2T-\nu/4T)\overline{X(\mu/2T-\nu/4T)}}{\sum_\nu \tilde{w}_\nu |X(\mu/2T-\nu/4T)|^2}$$

where

$$N(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(-2\pi ift)n(t)dt.$$

Taking into account the fact that by the present replacement of $\hat{C}_{yx}(\tau)$ and $\hat{C}_{xx}(\tau)$ by $C_{yx}(\tau)$ and $C_{xx}(\tau)$ the mean values of the denominator and numerator of the estimate of $A(f_\mu)$ are affected very little and the variances are reduced, we can see that our analysis of the bias in section 3 maintains its validity for our present new estimate, and further, that sampling variability of the new estimate will not exceed that of the former one. We have adopted $\hat{A}(f_\mu)$ in the stage of analysis of the sampling variability of our estimate because of its simplicity of statistical structure, but by our present argument we can see that $\hat{\hat{A}}(f_\mu)$ will be more stable than $\hat{A}(f_\mu)$ though the difference will actually be small. For practical applications, therefore we had better use $\hat{\hat{A}}(f_\mu)$ instead of $\hat{A}(f_\mu)$, and for evaluation of the sampling variability of $\hat{\hat{A}}(f_\mu)$ and its related quantities the results obtained for $\hat{A}(f_\mu)$ will be applied.

Usually in the practical computation of our estimate only the values of $C_{yx}(\tau)$ and $C_{xx}(\tau)$ at those values of τ which are integral multiples of some fixed constant $\Delta\tau$ are available. If $\Delta\tau$ is choosen to be small enough so that the powers of $x(t)$ and $y(t)$ at frequencies higher than $1/2\Delta\tau$ are negligibly small and

$$p_x(f) \gg \sum_{k\neq 0} p_x\left(f+\frac{k}{\Delta t}\right)$$

$$p_y(f) \gg \sum_{k\neq 0} p_y\left(f+\frac{k}{\Delta t}\right) \, *$$

hold in the range of f of our present concern, we can replace the integrals of $C_{yx}(\tau)$ and $C_{xx}(\tau)$ by the corresponding sums of $C_{xy}(l\Delta\tau)$ and $C_{xx}(l\Delta\tau)$, i.e.,

$$\int_{-T}^{T} \exp\left(2\pi i \frac{\mu}{2T}\tau\right) W(\tau) C_{yx}(\tau) d\tau$$

can be replaced by

$$\Delta\tau \sum_{l} \exp\left(-2\pi i \frac{\mu}{2T} l\Delta\tau\right) W(l\Delta\tau) C_{yx}(l\Delta\tau), **$$

and

$$\int_{-T}^{T} \exp\left(-2\pi i \frac{\mu}{2T}\tau\right) W(\tau) C_{xx}(\tau) d\tau$$

by

$$\Delta\tau \sum_{l} \exp\left(-2\pi i \frac{\mu}{2T} l\Delta\tau\right) W(l\Delta\tau) C_{xx}(l\Delta\tau).$$

For our trigonometric window we have

$$\begin{aligned} W(l\Delta\tau) &= \sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m} l\Delta\tau\right) && \text{when } |l\Delta\tau| < T_m \\ &= \frac{1}{2}\left(\sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m} l\Delta\tau\right)\right) && \text{when } l\Delta\tau = T_m \\ &= 0 && \text{otherwise,} \end{aligned}$$

---

* If the mecessary modification of the coherency is admitted, this condition may be replaced by the weaker one $|A(f)|^2 p_x(f) \gg \sum_{k\neq 0} |A(f+k/\Delta\tau)|^2 p_x(f+k/\Delta\tau)$.

** The summation $\sum_{l}$ is extended all over the l's satisfying $|l\Delta\tau| \leqq T$.

and we get our final computation scheme for the estimate $\hat{A}(f)(0 \leqq f \leqq 1/2\Delta\tau)$ of the frequency response function:

1) For $T_m = h\Delta\tau$ we calculate

$$\bar{p}_{yx}(f) = \Delta\tau \sum_{l=-h}^{h} \exp(-2\pi i f l \Delta\tau) C^*_{yx}(l\Delta\tau)$$

$$\bar{p}_{xx}(f) = \Delta\tau \sum_{l=-h}^{k} \exp(-2\pi i f l \Delta\tau) C^*_{xx}(l\Delta\tau)$$

where

$$C^*_{yx}(l\Delta\tau) = C_{yx}(l\Delta\tau) \qquad -h < l < h$$
$$= \frac{1}{2} C_{yx}(l\Delta\tau) \qquad l = \pm h$$
$$C^*_{xx}(l\Delta\tau) = C_{xx}(l\Delta\tau) \qquad -h < l < h$$
$$= \frac{1}{2} C_{xx}(l\Delta\tau) \qquad l = \pm h.$$

2) We smooth these $\bar{p}_{yx}(f)$ and $\bar{p}_{xx}(f)$ by the smoothing coefficient $\{a_n\}$ to obtain

$$\hat{p}_{yx}(f) = \sum_{n=-k}^{k} a_n \bar{p}_{yx}\left(f - \frac{n}{2T_m}\right)$$

$$\hat{p}_{xx}(f) = \sum_{n=-k}^{k} a_n \bar{p}_{xx}\left(f - \frac{n}{2T_m}\right).$$

3) The estimate $\hat{A}(f)$ is given by

$$\hat{A}(f) = \frac{\hat{p}_{yx}(f)}{\hat{p}_{xx}(f)}.$$

The estimate $\hat{\gamma}^2(f)$ of coherency $\gamma^2(f)$ is given by

$$\hat{\gamma}^2(f) = |\hat{A}(f)|^2 \frac{\hat{p}_{xx}(f)}{\hat{p}_{yy}(f)}$$

where $\hat{p}_{yy}(f)$ is obtained by replacing x by y in the definition of $\hat{p}_{xx}(f)$. The confidence region R for $A(f)$ can be obtained by putting these $\hat{A}(f)$ $\hat{p}_{xx}(f)$ and $\hat{p}_{yy}(f)$ into the formula given in the preceding section. The following approximations which are derived from that of R will also be useful.

$$P_r\left\{\left|\frac{|\hat{A}(f)| - |A(f)|}{|\hat{A}(f)|}\right| \leqq \sqrt{\frac{1}{n-1}\left(\frac{1}{\hat{\gamma}^2(f)} - 1\right) F(\delta, 2, 2(n-1))}\,,\right.$$
$$\left.|\hat{\phi}(f) - \phi(f)| \leqq \sin^{-1}\left(\sqrt{\frac{1}{n-1}\left(\frac{1}{\hat{\gamma}^2(f)} - 1\right) F(\delta, 2, 2(n-1))}\right)\right\} \geqq \delta$$

where

$$\hat{\phi}(f)=\arg \hat{A}(f)$$
$$\phi(f)=\arg A(f)$$
$$n=\text{the nearest integer to } \left(\frac{2T}{T_m}\right)\frac{1}{2\sum_{n=-k}^{k} a_n^2}$$

and $F(\delta, 2, 2(n-1))$ is given by the following relation for $F^2_{2(n-1)}$ which is distributed according to the F-distribution with d.f.s. 2 and $2(n-1)$

$$P_r\{F^2_{2(n-1)}\leqq F(\delta, 2, 2(n-1))\}=\delta,$$

and it is tacitly assumed that $\hat{\gamma}^2(f)$ and the quantity inside the square root are together positive and less than 1. When we use the shifted window $W_K(t)=W(t-K\Delta t)$ to compensate for the variation of phase shift, we usually adopt the following computing scheme

$$\hat{p}_{yx}(f)=\exp(-2\pi i f K\Delta\tau)\hat{p}_{yxK}(f)$$

where $\hat{p}_{yxK}(f)$ is obtained by replacing $C_{yx}(l\Delta\tau)$ by $C_{yx}((l+K)\Delta\tau)$ in the definition of $\hat{p}_{yx}(f)$.

In case where $|K\Delta\tau|$ is not very small compared with T_m, it will be more advisable to recalculate $C_{yx}(l\Delta\tau)$ by using $\{x(t), y(t+K\Delta\tau); -T\leqq t\leqq T\}$ in place of $\{x(t), y(t); -T\leqq t\leqq T\}$ or at least to replace the factor $1/2T$ in the definitions of $C_{xx}(\tau)$ and $C_{yx}(\tau)$ by the factor $1/(2T-|K\Delta\tau|)$. When the original data is given in the form $\{(x(n\Delta t), y(n\Delta t)); n=1, 2, \cdots, M\}$ and $\Delta\tau=m\Delta t$ (m; positive integer) we use in the above computation formulae those $C_{yx}(l\Delta t)$ and $C_{xx}(l\Delta\tau)$ defined by the following:

$$C_{yx}(l\Delta\tau)=\frac{1}{M}\sum_{n=1}^{M-lm} y((lm+n)\Delta t)x(n\Delta t) \qquad \text{when } l\geqq 0$$

$$=\frac{1}{M}\sum_{n=1-lm}^{M} y((lm+n)\Delta t)x(n\Delta t) \qquad \text{when } l<0$$

$$C_{xx}(l\Delta\tau)=\frac{1}{M}\sum_{n=1}^{M-|l|m} x((|l|m+n)\Delta t)x(n\Delta t).$$

When $\Delta\tau$ is sufficiently small as was assumed in the beginning of this section, the sampling variabilities of these estimates are considered to be almost the same as those of the estimates discussed in the previous sections.

As to the proper choice of the lag window $W(\tau)$ or $\{a_n\}$ the necessary informations are available in the preceding paper and in section 2 of this paper. One thing to be noted here is that, at those frequencies where coherencies are high, one should use $\{a_n\}$ of which the bandwidth

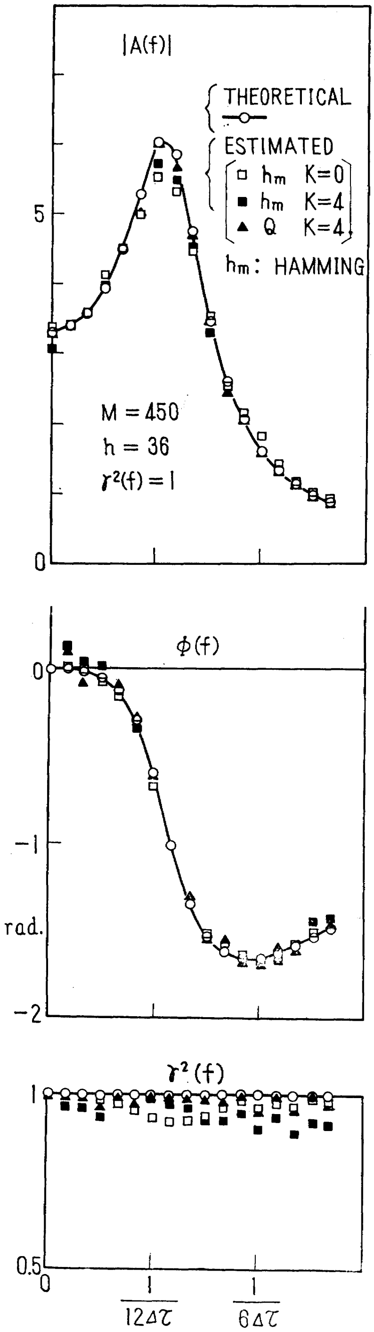


Fig. 3. Effect of the selection of windows. Perfectly coherent case.

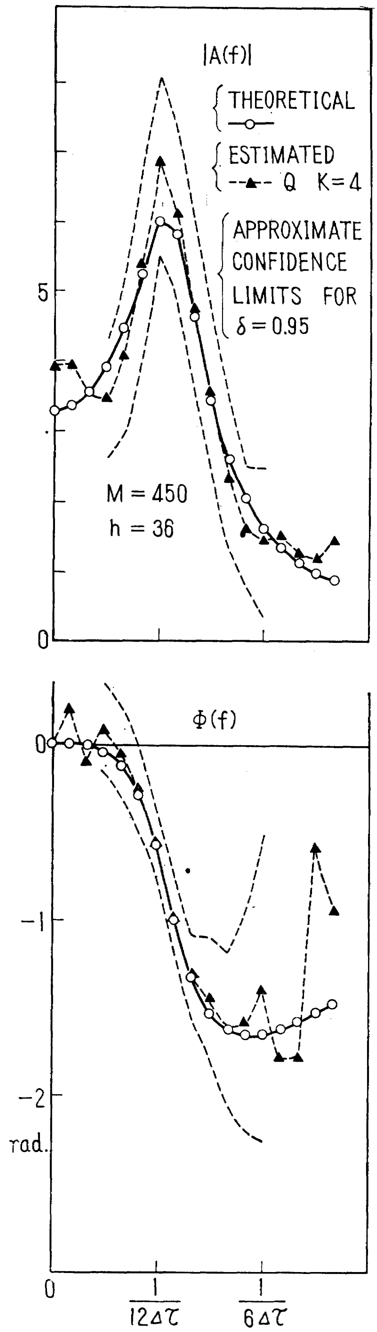
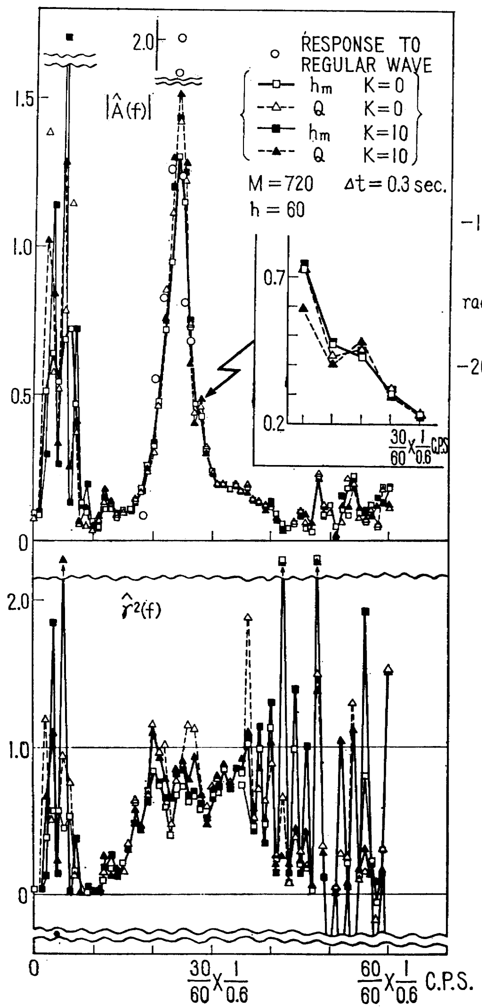


Fig. 4. Estimates of amplitude gain and phase shift with approximate 95% confidence limits. Necessary informations of coherency are given in Fig. 1 of § 4.



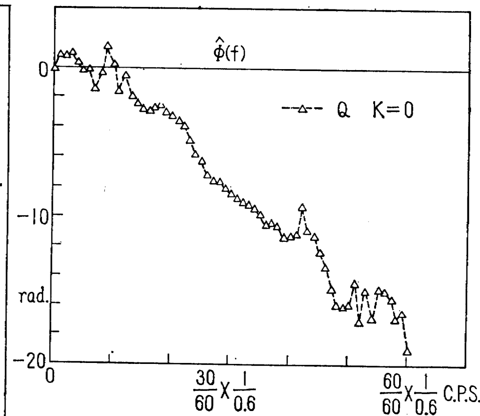


Fig. 5. Estimates of the amplitude gain and phase shift of a ship model, and the corresponding sample coherencies.

h_m stands for hamming.

defined in [1] is rather narrow.

6. Numerical examples

Figs. 3 and 4 give the results of applications of our statistical estimation procedure of the frequency response function, using various $\{a_n\}$ and K, to artificial time series. From these results, we can see that the approximations adopted in this paper do not impair the practical applicability of the results of our discussion. For instance, we can see that our estimates of the phase are fairly free from the bias due to smoothing, however, the gain suffers rather significant bias by the improper selection of the window.

The change of the phase shift in this artificial model is not very rapid, but in practical applications we often meet a system with phase shift varying much more rapidly and, correspondingly, with very sharp peak of the amplitude gain. In these circumstances, use of the properly shifted window with narrow bandwidth becomes more important to obtain a bias-free result. Fig. 5 gives the results of analysis of the response of a ship model. Here the input is the height of wave and the output

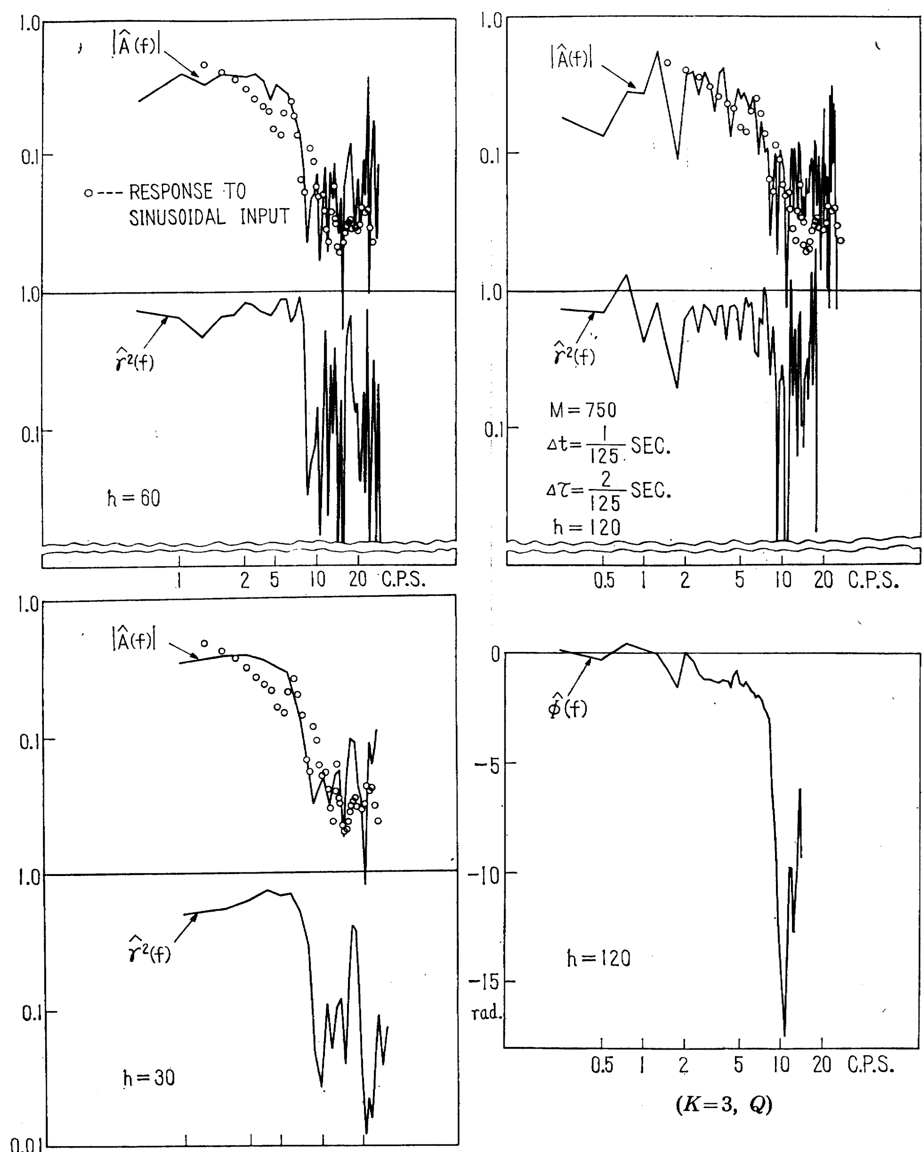


Fig. 6. Analysis of the linear relation between the front axle and the frame of an automobile. (by courtesy of Mr. I. Kanesige of the Isuzu Motor Company)

is the roll of the ship model. In this example we can see that the shifted windows produced more reliable estimates at frequencies ranging from $(20/60)\times(1/0.6)$ c.p.s. to $(30/60)\times(1/0.6)$ c.p.s.. The estimates of the power spectra of the input and output of these examples were given in the preceding paper [1, p. 20, Figs. 2, 3], and taking into account of the shape of the power spectrum of the input we can guess that some type of prewhitening operation is necessary to obtain reliable estimates in the wider range of frequencies. It seems that for this example our new window Q is better suited than that of hamming, and fairly large bias due to smoothing still remains at the peak of the amplitude gain.

Fig. 6 shows the results of the analysis, using Q, of the relation between the oscillation of the front axle and that of the frame of an automobile. We can see that the estimate of the amplitude gain thus obtained is in fairly good agreement with that obtained by the ordinary frequency response test only in the case $T_m=240\Delta t$. This is due to the existence of very sharp peaks around 9 c.p.s. and the estimate of the phase shift explains these circumstances more clearly. The shift $K\Delta\tau$ of the time axis of about 0.8 second will be necessary to get more reliable estimates at this frequency.

We have illustrated these two examples of ship model and automobile for the purpose of showing the complexity of practical problems. For practical applications, much more skillful use of the method will be necessary, but we believe that the use of well designed and properly shifted window will eventually lead to successful results.

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