# ON THE DESIGN OF LAG WINDOW FOR THE ESTIMATION OF SPECTRA

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## 1. Introduction and summary

In this paper we shall develop a design principle of the lag windows for the estimation of power spectra. In the estimation of a power spectral density function through the sample autocovariance function, it is inevitable to use some sort of smoothing to assure the stability of the estimate. Multiplication of the covariance function by a lag window means the smoothing or the averaging of the power spectral density, and if the window is not properly selected the smoothed spectrum may show a quite unreasonable appearance such as one indicating negative power. This fact sometimes caused trouble for the research workers who tried to perform the spectral analysis of stationary time series and the statisticians tried to develop a proper lag window which would evade this difficulty. It seems, to the present author, that at this stage of development of the method the main concern of the statisticians was the minimization of the undesirable side lobes of the Fourier-transform of the lag window or the spectral window. Knowing that some type of smoothing is inevitable in the estimation of the power spectral density function, some statisticians discussed consistency of the estimate. But, from the standpoint of practical application of the window, where the length of observation is limited, the consistency is not so important as the development of the window with desirable shape of the side lobes.

The windows which are used most often today in practical applications of the spectral analysis in many research fields will probably be the one called hamming, of which use was recommended early in 1949 by J. W. Tukey and R. W. Hamming [4]. Another window which is called by the name of hanning is also used very often. These windows were designed so as to minimize some of the undesirable features of the side lobes of the corresponding spectral windows as was discussed in the book by R. B. Blackmann and J. W. Tukey [3]. In this book it was also pointed out that the use of some type of prewhitening operation or rejection filtration is most effective to avoid this undesirable leakage of the power through the side lobes. In the recent paper, discussing the undamped oscillation of the sample autocovariance function, the present author gave some numerical examples which show little difference between hamming and hanning and it was concluded that further modifications of the windows would be unnecessary if the data were properly prewhitened. This conclusion is clearly correct if we focus our attention only on the leakage of the power through the side lobes but there still remains the averaging effect of the window.

In practical applications of the spectral method it is quite often that we have to analyse the spectrum with many sharp peaks and deep valleys, and in these cases, as the experimental study proceeds, the research workers usually become less satisfied with the knowledge of a rough averaged shape of the power spectral density function. Unfortunately in these circumstances the difficulties are not dissolved by the use of the prewhitening, and we come to recognize that the prewhitening and the rejection filtration are to be used to avoid the leakage of the power through the side lobes of the spectral window and the modification of the window is to be used to avoid the bias due to averaging. Thus we must have recourse to the modification of the spectral window again.

In this paper we shall introduce the notion of the bandwidth of a spectral window which is an index of the range over which the smoothing operation is effectively extended. With the definition of the bandwidth we develop the design of lag window as follows. First we restrict our window to a class and keep the estimates unbiased for the variation of the spectral density function which we shall assume to be locally approximated by some polynomial of the ordinate, then we seek for the window, which gives the minimum of the variance of estimate among those with a preassigned bandwidth.

Taking into account the computational ease and the convenience of the compatibility between the windows, we shall in this paper exclusively treat windows of trigonometric sum type and the above stated minimization is carried out within a class of windows represented by a trigonometric sum of some given order. Some windows are explicitly given, among them are windows which are bias free for the variation of the spectral density function represented by a polynomial up to the 2nd and 4th degree of the ordinate. The numerical results show that when we restrict our attention to the window of the 1st order and do not care for the bias due to the non-linear variation of the spectral density function the window which is obtained by our present design principle is very nearly equal to hamming and hanning, and, roughly speaking, it is situated between these two. Thus we can see that the performances of these two windows will be excellent in the range of frequencies where the spectral density function shows locally only a linear variation. Numerical comparisons of the windows are made and it is recommended to practically use the window which is of the 2nd order and is bias-free for up to the quadratic variation of the spectral density function. Some practical examples of applications are given of the cases where our present window yields higher values of estimates at the peaks and lower values at the valleys than the window like hamming. The results suggest that it will be most effective to apply many types of windows successively to one and the same problem to get an insight into the order of magnitude of bias due to smoothing. The use of our window for the estimation of the crossspectra or the frequency response function of a linear time-invariant system is discussed in [2].

## 2. An estimate of the spectral density function and its properties

In this section we shall analyse statistical properties of an estimate of the spectral density function. Usually there are discrepancies between the theoretical models adopted for evaluation of statistical properties of the estimate and the real problems to which the results are applied. For our present purpose proper approximations are sometimes more necessary and useful than the formal rigour. Throughout the present paper we shall assume that the process $x(t)$ is a real stationary Gaussian process with a bounded and continuous power spectral density function $p_x(f)$ and accordingly with zero-mean. Thus if we define

$$R(\tau)=Ex(t+\tau)x(t)$$

then we have

$$R(\tau)=\int_{-\infty}^{\infty} \exp(2\pi i f\tau)\, p_x(f)\, df\,.$$

It is also well known that $x(t)$ allows a representation

$$x(t)=\int_{-\infty}^{\infty} \exp(2\pi i f t)\, dZ(f)$$

where $Z(f)$ is a complex orthogonal process with $E|dZ(f)|^2=p_x(f)df$. Taking into account the fact that $x(t)$ is real, we have

$$x(t)=\int_{-\infty}^{\infty} \exp(-2\pi i f t)d\overline{Z(f)}.$$

It can be seen that under the present assumption of $x(t)$ all analytical operations which will be applied in this paper to functions of $x(t)$ including $x(t)$ itself are legitimate.

Given a sample $\{x(t)\,;-T\leqq t\leqq T\}$, we define for integral $\nu$'s

$$X\left(\frac{\nu}{2T}\right)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}t\right)x(t)dt\,.$$

Then we have

$$X\left(\frac{\nu}{2T}\right)=\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp\left(-2\pi i\left(\frac{\nu}{2T}-f\right)t\right)dt\right]dZ(f)$$
$$=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dZ(f)$$

where

$$W_T(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(-2\pi ift)dt.$$

It holds that

$$EX\left(\frac{\nu}{2T}\right)=0$$

$$E\left|X\left(\frac{\nu}{2T}\right)\right|^2=\int_{-\infty}^{\infty}\left|W_T\left(\frac{\nu}{2T}-f\right)\right|^2 p_x(f)df.$$

Since the relations

$$\int_{-\infty}^{\infty}|W_T(f)|^2df=1$$

$$\lim_{T\to\infty}\int_{-F}^{F}|W_T(f)|^2df=1 \qquad \text{for any } F>0$$

hold, we have

$$E\left|X\left(\frac{\nu}{2T}\right)\right|^2\sim p_x\left(\frac{\nu}{2T}\right)$$

where the symbol $\sim$ means that the both side members are nearly equal when $T$ is sufficiently large and $p_x(f)$ is smooth enough to be taken as a constant in the range of $f$ where $|W_T(\nu/2T-f)|^2p_x(f)$ shows a significant contribution to the integral. We shall hereafter use the symbol $\sim$ to designate the approximate equality between the both side members under the condition concerned.

Now we have

$$EX\left(\frac{\nu}{2T}\right)\overline{X\left(\frac{\mu}{2T}\right)}$$
$$=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)W_T\left(\frac{\mu}{2T}-f\right)p_x(f)df,$$

and

$$EX\left(\frac{\nu}{2T}\right)X\left(\frac{\mu}{2T}\right)=E\left[\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dZ(f)\int_{-\infty}^{\infty}W_T\left(\frac{\mu}{2T}-f'\right)dZ(f')\right]$$

$$=E\left[\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dZ(f)\int_{-\infty}^{\infty}W_T\left(\frac{\mu}{2T}+f''\right)d\overline{Z(f'')}\right]$$

$$=F\left[\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)dZ(f)\overline{\int_{-\infty}^{\infty}W_T\left(\frac{\mu}{2T}+f''\right)dZ(f'')}\right]$$

$$=\int_{-\infty}^{\infty}W_T\left(\frac{\nu}{2T}-f\right)W_T\left(-\frac{\mu}{2T}-f\right)p_x(f)df.$$

By assuming $p_x(f)$ to be nearly a constant in the range of $f$ where the contributions of $W_T(\nu/2T-f)W_T(\pm\mu/2T-f)p_x(f)$ to the integrals are significant, or by taking into account the relation

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}|W_T(f_1-f)W_T(f_2-f)|df=0 \qquad (f_1\neq f_2)$$

we have for $\nu\neq\pm\mu$

$$EX\left(\frac{\nu}{2T}\right)\overline{X}\left(\frac{\mu}{2T}\right)\sim 0$$

$$EX\left(\frac{\nu}{2T}\right)X\left(\frac{\mu}{2T}\right)\sim 0$$

and for $\nu\neq 0$

$$EX\left(\frac{\nu}{2T}\right)X\left(\frac{\nu}{2T}\right)\sim 0\,.$$

From these relations we can see that the real and imaginary parts of $X(\nu/2T)$'s $(\nu>0)$ can be considered to be uncorrelated and that the variance of the real part of each $X(\nu/2T)$ is equal to that of the imaginary part. Thus, in the following, when $T$ is sufficiently large to ensure the above stated approximations we can treat the real and imaginary parts of $X(\nu/2T)$'s $(\nu=0, 1, 2, \cdots)$ as mutually independent Gaussian random variables with zero-mean and variance $p_x(\nu/2T)/2$, with the exception of $X(0/2T)$ of which the variance of the real part is taken to be equal to $p_x(0)$ and the imaginary part is vanishing. Hereafter we shall assume the validity of this approximation. Now, we can see that $(p_x(\nu/2T)/2)^{-1}|X(\nu/2T)|^2$ is distributed as a $\chi^2$ random variable with d.f. 2 and $|X(\nu/2T)|^2$, an estimate of $p_x(\nu/2T)$, has coefficient of variation equal to 1.

We adopt here an averaging procedure with weight $\{w_\nu\}$ to get an estimate $I(\nu/2T)$ of $p_x(\nu/2T)$

$$I\left(\frac{\nu}{2T}\right)=\sum_{\mu} w_{\mu}\left|X\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right)\right|^{2}.$$

Hereafter the convergence of $\sum_{\mu}|w_{\mu}|$ is assumed. We have

$$EI\left(\frac{\nu}{2T}\right)\sim\sum_{\mu} w_{\mu}p_{x}\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right)$$

and for $\nu/2T>B$, a quantity to be defined later, we have

$$D^{2}I\left(\frac{\nu}{2T}\right)\sim\sum_{\mu}|w_{\mu}|^{2}p_{x}^{2}\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right).$$

For $\nu/2T\leqq B$ we have to modify the evaluation formula of $D^{2}I(\nu/2T)$ but for practical applications it will be sufficient to remember that $D^{2}I(0/2T)$ is approximately twice as large as that obtained by formally applying the present formula, and we shall disregard this case in the following discussion.

We give another representation of $|X(\nu/2T)|^{2}$,

$$\begin{aligned}\left|X\left(\frac{\nu}{2T}\right)\right|^{2}&=\left|\frac{1}{2T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}t\right)x(t)\,dt\right|^{2}\\&=\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}(t-s)\right)x(t)x(s)\,dt\,ds\\&=\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}\tau\right)C(\tau)\,d\tau\end{aligned}$$

where

$$C(\tau)=\frac{1}{2T}\int_{-T}^{T}x^{*}(\tau+s)x^{*}(s)ds$$

and $x^{*}(t)$ represents a periodic function with period $2T$ and identical with $x(t)$ in $(-T,\ T)$. Using this representation we get

$$\begin{aligned}I\left(\frac{\nu}{2T}\right)&=\sum_{\mu} w_{\mu}\left|X\left(\frac{\nu-\mu}{2T}\right)\right|^{2}\\&=\int_{-T}^{T}\exp\left(-2\pi i\frac{\nu}{2T}\tau\right)W(\tau)C(\tau)d\tau\end{aligned}$$

where

$$W(\tau)=\sum_{\mu} w_{\mu}\exp\left(2\pi i\frac{\mu}{2T}\tau\right).$$

This $W(\tau)$ is by definition the lag window corresponding to the spectral window $\{w_{\mu}\}$.

We shall now discuss the bias due to smoothing. We have seen that

$$EI\left(\frac{\nu}{2T}\right) \sim \sum_{\mu} w_{\mu} p_x\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right),$$

and if we can assume that $p_x(f)$ permits the approximation

$$p_x(f-\Delta f)=p_x(f)\left(\sum_{j=0}^{r} b_j(\Delta f)^j\right)$$

in the range of $\Delta f$ where $w_{\mu}p_x(f-\mu/2T)$ shows a significant contribution to the summation, then assuming the convergence of the infinite sums we have

$$\sum_{\mu} w_{\mu} p_x\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right)=p_x\left(\frac{\nu}{2T}\right)\left(\sum_{j=0}^{r} b_j \sum_{\mu}\left(\frac{\mu}{2T}\right)^j w_{\mu}\right).$$

As the approximation is local we shall here interpret the infinite sums in the sense of $(r+1)$st Cesàro mean $(C.\ r+1)$. Thus we get

$$(2\pi i)^j \sum_{\mu}\left(\frac{\mu}{2T}\right)^j w_{\mu}=\frac{d^j}{d\tau^j} W(\tau)\Big|_{\tau=0} \quad (C.\ r+1) \quad j=0,\ 1,\ 2,\cdots,\ r$$

if the derivatives exist, and consequently

$$\sum_{\mu} w_{\mu} p_x\left(\frac{\nu}{2T}-\frac{\mu}{2T}\right)=p_x\left(\frac{\nu}{2T}\right)\left(\sum_{j=0}^{r} b_j\left(\frac{1}{2\pi i}\right)^j \frac{d^j}{d\tau^j} W(0)\right) \quad (C.\ r+1).$$

From this last relation we can see that to keep our estimate unbiased we must use the lag window of which derivatives up to the $r$th order are all vanishing at the origin. We define the band-width $B$ of our spectral window by

$$B=2\left(\frac{\sum_{\mu}\left(\frac{\mu}{2T}\right)^2 |w_{\mu}|^2}{\sum_{\mu} |w_{\mu}|^2}\right)^{\frac{1}{2}}.$$

For evaluation of this $B$ we can conveniently use the relations

$$2T\sum_{\mu}\left(\frac{\mu}{2T}\right)^2 |w_{\mu}|^2=\left(\frac{1}{2\pi}\right)^2 \int_{-T}^{T}\left|\frac{d}{d\tau} W(\tau)\right|^2 d\tau$$

$$2T\sum_{\mu} |w_{\mu}|^2=\int_{-T}^{T} |W(\tau)|^2 d\tau.$$

In practical applications of our estimation procedure the integral in the definition of $I(\nu/2T)$ through $C(\tau)$ is replaced by a sum and we get

$$I_{\Delta t}\left(\frac{\nu}{2T}\right)=\sum_{l=-N}^{N}\exp\left(-2\pi i\frac{\nu}{2T}l\Delta t\right)W(l\Delta t)C(l\Delta t)\Delta t$$

where we have assumed $2T=(2N+1)\Delta t$. This assumption is made only for the ease of mathematical treatment, and for practical application $N$ is determined by the relation $(2N-1)\Delta t<2T\leqq(2N+1)\Delta t$.

As

$$W(\tau)C(\tau)=\sum_{\mu=-\infty}^{\infty}\exp\left(2\pi i\frac{\mu}{2T}\tau\right)I\left(\frac{\mu}{2T}\right)\frac{1}{2T}\quad \text{w.p. } 1$$

we have

$$\begin{aligned}I_{\Delta t}\left(\frac{\nu}{2T}\right)&=\sum_{\mu=-\infty}^{\infty}\left(\sum_{l=-N}^{N}\exp\left(-2\pi i\frac{\nu-\mu}{2T}l\Delta t\right)\left(\frac{\Delta t}{2T}\right)\right)I\left(\frac{\mu}{2T}\right)\\&=\sum_{m=-\infty}^{\infty}I\left(\frac{\nu+m(2N+1)}{2T}\right)\\&=\sum_{m=-\infty}^{\infty}I\left(\frac{\nu}{2T}+\frac{m}{\Delta t}\right)\quad \text{w.p. } 1\end{aligned}$$

and

$$\begin{aligned}EI_{\Delta t}\left(\frac{\nu}{2T}\right)&=\sum_{m=-\infty}^{\infty}EI\left(\frac{\nu}{2T}+\frac{m}{\Delta t}\right)\\&\sim\sum_{m=-\infty}^{\infty}\sum_{\mu}w_\mu p_x\left(\frac{\nu}{2T}-\frac{\mu}{2T}+\frac{m}{\Delta t}\right)\\&=\sum_{\mu}w_\mu\sum_{m=-\infty}^{\infty}p_x\left(\frac{\nu}{2T}+\frac{m}{\Delta t}-\frac{\mu}{2T}\right).\end{aligned}$$

To $I_{\Delta t}(\nu/2T)$ we can also give another expression

$$\begin{aligned}I_{\Delta t}\left(\frac{\nu}{2T}\right)&=\sum_{m=-\infty}^{\infty}I\left(\frac{1}{\Delta t}\left(\frac{\nu}{2N+1}+m\right)\right)\\&=\sum_{m=-\infty}^{\infty}\sum_{\mu}w_\mu\left|X\left(\frac{\nu-\mu+m(2N+1)}{2T}\right)\right|^2\\&=\sum_{m=-\infty}^{\infty}\sum_{\mu'}\left|X\left(\frac{\nu-\mu'}{2T}\right)\right|^2 w_{\mu'+m(2N+1)}\\&=\sum_{\mu'}\left|X\left(\frac{\nu-\mu'}{2T}\right)\right|^2\sum_{m=-\infty}^{\infty}w_{\mu'+m(2N+1)},\end{aligned}$$

hence, when we put $w'_\mu=\sum_{m=-\infty}^{\infty}w_{\mu+m(2N+1)}$

$$I_{\Delta t}\left(\frac{\nu}{2T}\right)=\sum_{\mu}w'_\mu\left|X\left(\frac{\nu-\mu}{2T}\right)\right|^2.$$

Using this expression we have

$$EI_{\Delta t}\left(\frac{\nu}{2T}\right) \sim \sum_{\mu} w'_{\mu} p_x\left(\frac{\nu-\mu}{2T}\right)$$

$$D^2 I_{\Delta t}\left(\frac{\nu}{2T}\right) \sim \sum_{\mu} (w'_{\mu})^2 p_x^2\left(\frac{\nu-\mu}{2T}\right).$$

It should be noted that

$$w'_{\mu+2N+1} = w'_{\mu}$$

holds and we can use $I_{\Delta t}(\nu/2T)$ as an estimate of $p_x(\nu/2T)$ only under the condition that the sum of the values of $p_x(f)$ at the frequencies $(\nu/2T)+(m/\Delta t)$ $(m=\pm 1, \pm 2, \cdots)$ are negligibly small compared with $p_x(\nu/2T)$. Along with this condition we shall further assume hereafter that the bandwidth $B$ of the spectral window is sufficiently narrow compared with $1/\Delta t$ so that the sum of $w_{\mu}$'s $(|\mu|>N)$ are taken to be negligible.

If these conditions hold, then the difference between $I_{\Delta t}(\nu/2T)$ and $I(\nu/2T)$ is negligibly small in the sense of mean square and the former formulae for the bias and the bandwidth of $I(\nu/2T)$ with $W(\tau)$ and $(d^j/d\tau^j)W(\tau)$'s can be used for evaluation of these quantities of $I_{\Delta t}(\nu/2T)$. However, in this case there remains arbitrariness of the value of $W(\tau)$ at $\tau \neq l\Delta t$ and if we want to utilize the former evaluation formulae we have to restrict our $W(\tau)$ to those which are very nearly the one with the minimum $B$ among the lag windows giving the same values $W(l\Delta t)$ at $\tau = l\Delta t$ $(l=-N, \cdots, 0, \cdots, N)$. This suggests that in the evaluation formulae we should use $W(\tau)$ which is almost equal to the polygon obtained by linearly interpolating the values of $W(\tau)$ at $\tau = l\Delta t$. These considerations combined with the computational ease lead to the following definition of the lag window of trigonometric sum type and its necessary evaluation formulae. The lag window $W(\tau)$ is called of trigonometric sum type of the $k$th order when it is defined by

$$\begin{aligned}
W(\tau) &= \sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m}\tau\right) && \text{for } |\tau| < T_m - \Delta t \\
&= \frac{1}{2}\sum_{n=-k}^{k} a_n \exp\left(2\pi i \frac{n}{2T_m}\tau\right) && \text{for } \tau = \pm T_m \\
&= 0 && \text{for } |\tau| > T_m + \Delta t \\
&= \frac{T_m - |\tau|}{\Delta t}[W(T_m - \Delta t) - W(T_m)] + W(T_m) && \text{for } T_m - \Delta t \leqq |\tau| < T_m \\
&= \frac{T_m + \Delta t - |\tau|}{\Delta t} W(T_m) && \text{for } T_m < |\tau| \leqq T_m + \Delta t
\end{aligned}$$

where $\Delta t \geqq 0$, $a_n$'s are real and $a_{-n}=a_n$. We shall hereafter assume

$$T_m = h\Delta t.$$

For the present type of window we have

$$I_{\Delta t}\left(\frac{\nu}{2T}\right)=\sum_{l=-N}^{N}\exp\left(-2\pi i\frac{\nu}{2T}l\Delta t\right)W(l\Delta t)C(l\Delta t)$$
$$=\sum_{n=-k}^{k}a_n\hat{I}_{\Delta t}\left(\frac{\nu}{2T}-\frac{n}{2h\Delta t}\right)$$

where

$$\hat{I}_{\Delta t}\left(\frac{\nu}{2T}-\frac{n}{2h\Delta t}\right)=\sum_{l=-h}^{h}\exp\left(-2\pi i\left(\frac{\nu}{2T}-\frac{n}{2h\Delta t}\right)l\Delta t\right)C^*(l\Delta t)$$

and

$$C^*(l\Delta t)=C(l\Delta t) \qquad \text{for } l=0,\ \pm 1,\ \pm 2,\cdots,\ \pm(h-1)$$
$$C^*(l\Delta t)=\frac{1}{2}C(l\Delta t) \qquad \text{for } l=\pm h.$$

From this expression of $I_{\Delta t}(\nu/2T)$ we can see that once the values $\{\hat{I}_{\Delta t}(\nu/2T-n/2h\Delta t)\}$ are given we can very easily compare the effects of windows by simply changing the coefficients $\{a_n\}$. This compatibility, of which we have mentioned in the introduction of this paper, of the windows of trigonometric sum type is one of the most important features of the windows of this type. It seems more or less obvious that under the assumption of sufficiently large $T$ the value $I_{\Delta t}(f)$, which is obtained by replacing $\nu/2T$ by an arbitrary $f$ in the definition of $I_{\Delta t}(\nu/2T)$, will give an estimate of $p_x(f)$ and its expection and variance are obtained by simply replacing $\nu/2T$ by $f$ in the corresponding formulae for $I_{\Delta t}(\nu/2T)$. This statement can most conveniently be verified by representing $X(f)=\frac{1}{\sqrt{2T}}\int_{-T}^{T}\exp(2\pi ift)x(t)dt$ in terms of $X(\mu/2T)$'s. It is seen that when $f$ is not an integral multiple of $1/2T$ the dependence between the real parts and between the imaginary parts of $X(f+(\nu/2T))$ and $X(f)$ are induced by the existence of the power of $x(t)$ at the very low frequencies. This is the reason why we have developed our evaluation formula for $X(\nu/2T)$ and we shall not go here into the details of $X(f)$.

In passing we note that if we want to know the values of $I_{\Delta t}(f)$ at $f=m/2h\Delta t$ $(m=0,\ \pm 1,\ \pm 2,\ \cdots,\ \pm h)$ we can use the relation

$$I_{\Delta t}\left(\frac{m}{2h\Delta t}\right)=\sum_{n=-k}^{k}a_n\hat{I}_{\Delta t}\left(\frac{m-n}{2h\Delta t}\right)$$

and our estimates $\{I_{\Delta t}(m/2h\Delta t)\}$ are obtained from $\{\hat{I}_{\Delta t}(m/2h\Delta t)\}$ by simply applying the running average with the weight $\{a_n\}$. Taking into account that $\frac{d}{d\tau}\left(\sum_{\nu=-k}^{k} a_\nu \exp\left(2\pi i \frac{\nu}{2T_m}\tau\right)\right)_{\tau=Tm}=0$ and that we are concerned only with those windows for which $|W(T_m)|$ is small, we obtain the following evaluation formulae

$$\left(\frac{1}{2\pi}\right)^2\int_{-T}^{T}\left|\frac{d}{d\tau}W(\tau)\right|^2 d\tau=2T_m\sum_{n=-k}^{k}\left(\frac{n}{2T_m}\right)^2|a_n|^2+\left(\frac{1}{2\pi}\right)\frac{4}{\Delta t}(W(T_m))^2+o(\Delta t)$$

$$\int_{-T}^{T}|W(\tau)|^2 d\tau=2T_m\sum_{n=-k}^{k}|a_m|^2+o(\Delta t).$$

Therefore we get

$$B=\frac{1}{2T_m}2b+o(\Delta t)$$

where $b$ is by definition

$$b=\left(\frac{\sum_{n=-k}^{k} n^2|a_n|^2+\frac{2h}{(2\pi)^2}\left(\sum_{n=-k}^{k}(-1)^n a_n\right)^2}{\sum_{n=-k}^{k}|a_n|^2}\right)^{\frac{1}{2}}.$$

Now we shall proceed to evaluation of the variance of our estimate $I(\nu/2T)$.

Our present result of evaluation of the band width $B$ shows that $B$ is inversely proportional to $T_m$ or $h$, and in practical applications of the estimation procedure $B$ is usually kept so small as to assure the validity of approximating $p_x(f-\Delta f)$ locally by a polynomial of $\Delta f$ of rather low degrees $r$ and of assuming the variation of $p_x(f)$ to be small. In this case, using the polynomial approximation we can proceed entirely in the same way as in the case of evaluation of the bias of $I(\nu/2T)$, but taking into account that the variation of $p_x(f)$ is small, and that the range of $\mu$ for the significant values of $w_\mu^2$ is narrower than that of $\mu$ for the significant values of $w_\mu$, we adopt here the simplest approximation of 0th order or $p_x(\nu/2T-\mu/2T)=p_x(\nu/2T)$ in the summation formula and obtain

$$D^2I\left(\frac{\nu}{2T}\right)\sim\left(\sum_{\mu}|w_\mu|^2\right)p_x^2\left(\frac{\nu}{2T}\right)=\left[\frac{1}{2T}\int_{-T}^{T}|W(\tau)|^2 d\tau\right]p_x^2\left(\frac{\nu}{2T}\right)$$
$$=\left(\frac{2T_m}{2T}\sum_{\nu=-k}^{k}|a_\nu|^2\right)p_x^2\left(\frac{\nu}{2T}\right)+o(\Delta t).$$

Assuming the unbiasedness of our estimate, we have

$$C.V.I\left(\frac{\nu}{2T}\right)=\left(\frac{D^2\left(I\left(\frac{\nu}{2T}\right)\right)}{\left[E\left(I\left(\frac{\nu}{2T}\right)\right)\right]^2}\right)^{\frac{1}{2}}\sim\left(\frac{T_m}{2T}\left(2\sum_{\nu=-k}^{k}|a_\nu|^2\right)\right)^{\frac{1}{2}}.$$

We shall use this last quantity for evaluation of the variability of our estimate.

## 3. Design principle of the lag window

In this section using the results of the preceding section we shall develop a design principle of the lag window of trigonometric sum type. In practical applications of our estimation procedure an upper bound $B_0$ of $B$ in the frequency range of our concern is given from technical consideration. Then, assuming the type of the window to be predetermined, we select a $T_m$ for which $(1/T_m)b=B_0$ holds. In this case the relative accuracy of our estimate obtained by using the record of observation of length $2T$ is evaluated as

$$\left(\frac{T_m}{2T}\left(2\sum_{\nu=-k}^{k}|a_\nu|^2\right)\right)^{\frac{1}{2}}=\left(\frac{1}{T}\frac{1}{B_0}b\left(\sum_{\nu=-k}^{k}|a_\nu|^2\right)\right)^{\frac{1}{2}},$$

and we can see that for a given $B_0$ or $T_m$ the window which gives the minimum of $b(\sum_\nu|a_\nu|^2)$ is the best one in the sense of our evaluation formula. The values of $C(\tau)$ are only observed at integral multiples of some $\Delta t$ and we have already seen that when we put $T_m=h\Delta t$ the corresponding value of $b$ is given as a function of $h$ and $a_\nu$'s and is independent of $\Delta t$. Thus, taking into consideration the result of analysis of bias in the former section, our design principle of the optimum lag window is stated as follows. We calculate the sets of $a_\nu$'s which give under the condition of some local unbiasedness, the minimums, of $b(\sum_\nu|a_\nu|^2)$ for some values of $h$, and for practical applications we use the lag window determined by $T_m=h\Delta t$ and the set of $a_\nu$'s of which $b(h\Delta t)^{-1}$ is approximately equal to the given $B_0$.

For example, if we are to design for $h=36$ an optimum lag window which is of the 2nd order type and free from biases up to the 2nd order, we have only to find the set of $a_\nu$'s$(\nu=0, 1, 2)$ which satisfies the simultaneous linear equations

$$\left.\begin{aligned}a_{-1}&=a_1\\ a_{-2}&=a_2\\ \sum_{\nu=-2}^{2}a_\nu&=1\\ \sum_{\nu=-2}^{2}\nu^2a_\nu&=0\end{aligned}\right\}\qquad\text{(I)}$$

and minimizes the function

$$\left(\sum_{\nu=-2}^{2} a_\nu^2\right)\left(\sum_{\nu=-2}^{2}\nu^2 a_\nu^2+\frac{2h}{(2\pi)^2}\left(\sum_{\nu=-2}^{2}(-1)^\nu a_\nu\right)^2\right) \tag{II}$$

for $h=36$. From (I) we have

$$a_1=\frac{4}{6}(1-a_0)$$

$$a_2=-\frac{1}{6}(1-a_0)$$

and by substituting these results into (II) we have only to find the value of $a_0$ which minimizes

$$\left[a_0^2+2\left(\left(\frac{4}{6}\right)^2+\left(-\frac{1}{6}\right)^2\right)(1-a_0)^2\right]\left[2\left(\left(\frac{4}{6}\right)^2+2^2\left(-\frac{1}{6}\right)^2\right)(1-a_0)^2 \right.$$
$$\left. +C\left(\frac{1}{3}\right)^2(8a_0-5)^2\right]$$

where $C=2h(2\pi)^{-2}$ and $h=36$. We can easily find the value of this $a_0$ by following the ordinary minimization procedure.

As will be seen the value of $b$ of the optimum window remains nearly as a constant for each type of the window irrespective of the choice of the value of $h$ if at least $h$ lies in the range of our concern, this suggests practical applicability of our design principle.

In the following section we shall numerically discuss windows obtained by our present design principle.

## 4. Numerical analysis of the window

In this section we shall first illustrate some of the numerical results concerning the lag windows which are obtained by our present design procedure and then discuss the relative merits of windows to get a new type of window to recommend. In our experiences, up to the present time, of numerical computations of estimates of spectral density functions using hamming or hanning window we have most often encountered the cases where the value of $h$ in the range from 18 to 72 was considered to be adequate for the purpose of analysis. Accordingly, we treat here the cases where the values of $C=2h(2\pi)^{-2}$ are 1, 2, 3 and 4. These values correspond approximately to the values 20, 40, 60 and 80 of $h$ respectively.

For the sake of convenience we shall use the abbreviated notation $W_k(b, *)$ to designate the window which satisfies the following conditions :

1) it is of $k$th order type,
2) free from biases of even orders up to the $b$th order and
3) minimizes the quantity $*$, where $*$ designates any one of $\alpha$, $\alpha\beta$ and $\alpha^{-1}\beta$, with $\alpha$, $\beta$ defined by

$$\alpha=\sum_{n=-k}^{k} a_n^2$$

and

$$\beta=\alpha b^2=\sum_{n=-k}^{k} n^2|a_n|^2+C(\sum_{n=-k}^{k}(-1)^n a_n)^2.$$

It should be noted that all windows we treat here are symmetric around the $y$-axis, consequently are locally free from all the biases of odd orders. The coefficients $\{a_n\}$ which define $W_k(2(k-1), \alpha\beta)$, optimum windows in case $b=2(k-1)$ by our design principle, are given in Table 1 for different values of $C$. The shapes of these lag windows and their spectral windows are illustrated in Fig. 1. In this figure the windows are illustrated assuming $\Delta t=0$. For practical applications only the values of $W(t)$ at $t=l\Delta t$ ($l=0, \pm 1, \pm 2,\cdots, \pm h$) are necessary and in this sense, disregarding the values of $\Delta t$ and $h$, we shall hereafter identify the window with the set of coefficients $\{a_n\}$. As can be seen from the following discussion such an identification will cause little trouble when using the windows treated in this paper. In Table 1 the windows given in Blackman and Tukey [3] are also shown.

TABLE 1

| | $W_1(0, \alpha\beta)$ | | | | $W_2(2, \alpha\beta)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $C$ | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| $a_0$ | 0.5363 | 0.5192 | 0.5132 | 0.5100 | 0.6652 | 0.6466 | 0.6398 | 0.6364 |
| $a_1=a_{-1}$ | 0.2319 | 0.2404 | 0.2434 | 0.2450 | 0.2232 | 0.2356 | 0.2401 | 0.2424 |
| $a_2=a_{-2}$ | * | * | * | * | −0.0558 | −0.0589 | −0.0600 | −0.0606 |

| | $W_3(4, \alpha\beta)$ | | | | | hamming | hamming | 4th pair |
|---|---|---|---|---|---|---|---|---|
| $C$ | 1 | 2 | 3 | 4 | | * | * | * |
| $a_0$ | 0.7285 | 0.7098 | 0.7029 | 0.6992 | | 0.5400 | 0.5000 | 0.4200 |
| $a_1=a_{-1}$ | 0.2036 | 0.2176 | 0.2228 | 0.2256 | | 0.2300 | 0.2500 | 0.2500 |
| $a_2=a_{-2}$ | −0.0814 | −0.0870 | −0.0891 | −0.0902 | | * | * | 0.0400 |
| $a_3=a_{-3}$ | 0.0136 | 0.0145 | 0.0149 | 0.0150 | | * | * | * |

We can see that the shape of the window is affected very little by change of the value of $C$ or $h$, which enables us to practically use any

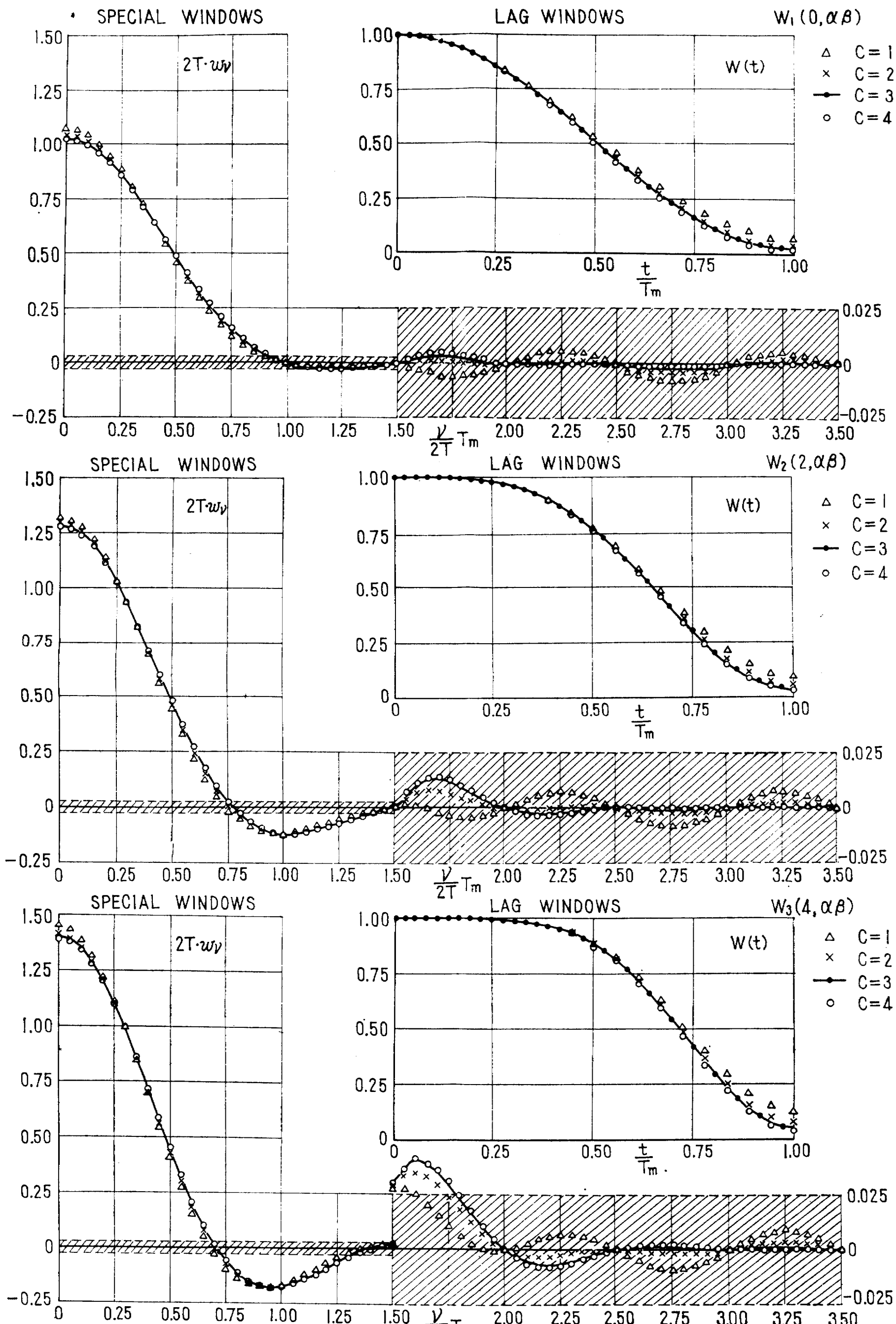


Fig. 1. Lag windows and spectral windows for various values of $C$.

window which is approximately equal to one of these windows for all the values of $h$ in the range treated here. The window $W_2(2,\ \alpha\beta)$ shows a significant change in shape from that of $W_1(0,\ \alpha\beta)$ and the characteristic feature of its spectral window is rather big negative first side lobes.

Besides the design principle stated in the preceding section there may be many others. For instance, the minimization principle of the variance of the estimate or of the bandwidth of the window. These principles lead to the windows $W_k(b,\ \alpha)$ and $W_k(b,\ \alpha^{-1}\beta)$ respectively. Considering that the shape of the window $W_k(2(k-1),\ \alpha\beta)$ was not sensitive to the change of $C$ we made numerical comparison of these windows $W_k(b,\ \alpha)$ $W_k(b,\ \alpha^{-1}\beta)$ and $W_k(b,\ \alpha\beta)$ for the case of $C=3$. The results are shown in table 2. In the last three rows are given the values $A_1(D)$ and $A_2(D)$ and $A_4(D)$ which are defined by the formulae at the margin of the table 2 to show the relative heights to the original densities at $f=0$, of the smoothed spectral densities obtained by replacing $R(\tau)=e^{-\lambda|\tau|}(\lambda>0)$ by $W(\tau)R(\tau)$, $W(\tau)R(\tau)$ being, by definition,

$$\begin{aligned} W(\tau)R(\tau) &= \sum_{\nu=-k}^{k} a_\nu \exp\left(2\pi i \frac{\nu}{2T_m}\tau\right)R(\tau) \qquad &|\tau|<T_m \\ &=0 &|\tau|>T_m\,. \end{aligned}$$

The values $A_1(D)$ $A_2(D)$ and $A_4(D)$ correspond to the cases where $1/2T_m$ is equal to $\lambda/2\pi$, $\lambda/4\pi$ and $\lambda/8\pi$ respectively, $\lambda/2\pi$ being the frequency with which $p(\lambda/2\pi)=p(0)/2$ holds. By inspecting Table 2 we can see that the increase of $k$ produces the decrease of $\alpha$ and the increase of $\alpha^{-1}\beta$ for $W_k(b,\ \alpha)$ and $W_k(b,\ \alpha\beta)$ when $b$ is kept constant. In practical applications it is more often that finer resolvability of our estimate is desired and we may conclude that unnecessary increase of $k$ is not profitable. From this conclusion we can realize the reason why the *Fourth Pair* ($a_0=0.42$ $a_1=a_{-1}=0.25$ $a_2=a_{-2}=0.04$) of Blackmann and Tukey [3], which is very nearly equal to the present $W_2(0,\ \alpha\beta)$, was not seriously recommended therein. Likewise, due to its wide bandwidth, the window $W_k(b,\ \alpha)$ is not recommendable for ordinary use. As to the windows $W_k(b,\ \alpha\beta)$ treated here, we may restrict our attention to $W_1(0,\ \alpha\beta)$, $W_2(2,\ \alpha\beta)$ and $W_3(4,\ \alpha\beta)$. Further, from table 2, we can see that $W_1(0,\ \alpha\beta)$, $W_2(2,\ \alpha\beta)$ and $W_3(4,\ \alpha\beta)$ are very nearly equal to $W_1(0,\ \alpha^{-1}\beta)$, $W_2(2,\ \alpha^{-1}\beta)$ and $W_3(4,\ \alpha^{-1}\beta)$ respectively, and we can conclude that in each case our window $W_k(2(k-1),\ \alpha\beta)$ has bandwidth very nearly equal to the narrowest possible value. Thus we may conclude that the window of the type $W_k(2(k-1),\ \alpha\beta)$ will be the most appropriate for ordinary use if the value of $k$ is properly chosen. From the values of $A_\nu(D)$ in the last three columns of Table 2 we can see that even though the value of $\alpha\beta$ is smaller for $W_1(0,\ \alpha\beta)$ than for $W_2(2,\ \alpha\beta)$, we have, for

$W_1(0, \alpha\beta)$, to adopt $T_m$ nearly twice as long as that for $W_2(2, \alpha\beta)$ to keep the bias $A(D)$ of the same order. Taking into account the computing effort, we can realize that this gives a significant advantage of $W_2(2, \alpha\beta)$ over $W_1(0, \alpha\beta)$. The advantage of $W_3(4, \alpha\beta)$ over $W_2(2, \alpha\beta)$ is not so significant as that of $W_2(2, \alpha\beta)$ over $W_1(0, \alpha\beta)$. Considering also the computing effort for $W_3(4, \alpha\beta)$, we can draw our final conclusion: *the window $W_2(2, \alpha\beta)$ will be the most recommendable for ordinary use.* We note here that the two very big negative side lobes of $W_2(2, \alpha\beta)$ are not harmful for ordinary applications and that, as can be seen from the shape of the spectral window of $W_2(2, \alpha\beta)$, the leakage of the power through the minor side lobes are rather less for $W_2(2, \alpha\beta)$ than for $W_1(0, \alpha\beta)$.

As slight changes of the coefficients $a_\nu$ for the convenience of computation are permissible, we introduce a window $Q$ which is a modification of $W_2(2, \alpha\beta)$ and is defined by

$$Q: \quad a_0=0.64, \quad a_1=a_{-1}=0.24, \quad a_2=a_{-2}=-0.06.$$

This window $Q$ is generally recommended for the estimation of the power spectrum, of the cross spectra and of the frequency response function of a linear time-invariant system. We shall treat this last problem in the separate paper [2].

## 5. Some remarks for the practical application of the estimation procedure and some numerical examples

In the definition of our estimate $I(f)$

$$I(f)=\int_{-T}^{T}\exp(-2\pi i f\tau)\,W(\tau)C(\tau)\,d\tau$$

we have used $C(\tau)$ which is given by

$$C(\tau)=\frac{1}{2T}\int_{-T}^{T-|T|}x(t+|\tau|)x(t)dt+\frac{1}{2T}\int_{T-|\tau|}^{T}x(t+|\tau|-2T)x(t)dt.$$

Usually $T$ is so large that $R(\tau)=Ex(t+\tau)x(t)$ can be considered to be vanishing for $|\tau|\geqq T$ and the expectation of the second member of the right hand side of this expression can be equated to zero. Therefore, if one wants to keep $C(\tau)$ unbiased as an estimate of $R(\tau)$ one should replace the factor $1/2T$ by $1/(2T-|\tau|)$ in the definition of $C(\tau)$ but if we consider the effect of $W(\tau)$ we can see that the effect of this correction is usually negligibly small. Further, if we can assume that $R(\tau)$ is considered to be negligibly small for $|\tau|\geqq 2T-3T_m$, then the covariance between the first and the second member of the right hand side of $C(\tau)$

TABLE

| | $W_1(0, \alpha)$ | $W_1\left(0, \frac{\beta}{\alpha}\right)$ | $W_1(0, \alpha\beta)$ | $W_2(0, \alpha)$ |
|---|---|---|---|---|
| $a_0$ | 0.3333 | 0.5272 | 0.5132 | 0.2000 |
| $a_1=a_{-1}$ | 0.3333 | 0.2364 | 0.2434 | 0.2000 |
| $a_2=a_{-2}$ | * | * | * | 0.2000 |
| $a_3=a_{-3}$ | * | * | * | * |
| $\sqrt{2\alpha}$ | 0.8165 | 0.8828 | 0.8740 | 0.6325 |
| $2\sqrt{\beta/\alpha}$ | 2.5820 | 1.1128 | 1.1254 | 3.2249 |
| $2\alpha\cdot 2\sqrt{\beta/\alpha}$ | 1.7213 | 0.8673 | 0.8596 | 1.2900 |
| $A_1(D)$ | 0.6666 | 0.7510 | 0.7449 | 0.4765 |
| $A_2(D)$ | 0.8670 | 0.9052 | 0.9024 | 0.7199 |
| $A_4(D)$ | 0.9608 | 0.9722 | 0.9714 | 0.8965 |

| | $W_2(2, \alpha)$ | $W_2\left(2, \frac{\beta}{\alpha}\right)$ | $W_2(2, \alpha\beta)$ | $W_3(2, \alpha)$ |
|---|---|---|---|---|
| $a_0$ | 0.4857 | 0.6475 | 0.6398 | 0.3333 |
| $a_1=a_{-1}$ | 0.3429 | 0.2350 | 0.2401 | 0.2857 |
| $a_2=a_{-2}$ | −0.0857 | −0.0588 | —0.0600 | 0.1429 |
| $a_3=a_{-3}$ | * | * | * | −0.0952 |
| $2\sqrt{2\alpha}$ | 0.9856 | 1.0360 | 1.0315 | 0.8165 |
| $2\sqrt{\beta/\alpha}$ | 2.4138 | 1.0536 | 1.0578 | 2.8140 |
| $2\alpha\cdot 2\sqrt{\beta/\alpha}$ | 2.3450 | 1.1307 | 1.1255 | 1.8760 |
| $A_1(D)$ | 0.7896 | 0.8422 | 0.8397 | 0.6518 |
| $A_2(D)$ | 0.9489 | 0.9643 | 0.9636 | 0.8746 |
| $A_4(D)$ | 0.9940 | 0.9958 | 0.9958 | 0.9779 |

2

| $W_2\left(0, \frac{\beta}{\alpha}\right)$ | $W_2(0, \alpha\beta)$ | $W_3(0, \alpha)$ | $W_3\left(0, \frac{\beta}{\alpha}\right)$ | $W_3(0, \alpha\beta)$ |
|---|---|---|---|---|
| 0.6202 | 0.4282 | 0.1429 | 0.6129 | 0.4229 |
| 0.2358 | 0.2433 | 0.1429 | 0.2182 | 0.2124 |
| −0.0459 | 0.0426 | 0.1429 | −0.0431 | 0.0434 |
| * | * | 0.1429 | 0.0184 | 0.0327 |
| 1.0000 | 0.7814 | 0.5346 | 0.9749 | 0.7416 |
| 1.0496 | 1.3296 | 4.2094 | 1.0247 | 1.3521 |
| 1.0496 | 0.8119 | 1.2026 | 0.9740 | 0.7438 |
| 0.8218 | 0.6798 | 0.3703 | 0.8014 | 0.6496 |
| 0.9512 | 0.8599 | 0.6024 | 0.9298 | 0.8261 |
| 0.9906 | 0.9543 | 0.8234 | 0.9782 | 0.9340 |

| $W_3\left(2, \frac{\beta}{\alpha}\right)$ | $W_3(2, \alpha\beta)$ | $W_3(4, \alpha)$ | $W_3\left(4, \frac{\beta}{\alpha}\right)$ | $W_3(4, \alpha\beta)$ |
|---|---|---|---|---|
| 0.6643 | 0.5571 | 0.5671 | 0.7085 | 0.7029 |
| 0.2306 | 0.2610 | 0.3247 | 0.2186 | 0.2228 |
| −0.0669 | −0.0190 | −0.1299 | −0.0875 | −0.0891 |
| 0.0041 | −0.0205 | 0.0216 | 0.0146 | 0.0149 |
| 1.0554 | 0.9468 | 1.0650 | 1.1075 | 1.1043 |
| 1.0508 | 1.1612 | 2.3746 | 1.0658 | 1.0718 |
| 1.1698 | 1.0409 | 2.6933 | 1.3073 | 1.3071 |
| 0.8514 | 0.8021 | 0.8361 | 0.8755 | 0.8740 |
| 0.9685 | 0.9428 | 0.9702 | 0.9793 | 0.9790 |
| 0.9966 | 0.9918 | 0.9981 | 0.9987 | 0.9988 |

$$A_\nu(D) = a_0(1-e^{-\nu\pi}) + 2\sum_{n=1}^{k} a_n \frac{1 - e^{-\nu\pi}\sqrt{1+\left(\frac{n}{\nu}\right)^2}\cos\left(n\pi + \tan^{-1}\frac{n}{\nu}\right)}{1+\left(\frac{n}{\nu}\right)^2}$$

$\nu = 1,\ 2,\ 4.$

can be considered to be vanishing. Thus we can see that this second member merely introduces the increase of the variance of $I(f)$ and for practical applications we should replace $C(\tau)$ by $\frac{1}{2T}\int_{-T}^{T-|\tau|} x(t+|\tau|)x(t)dt$.

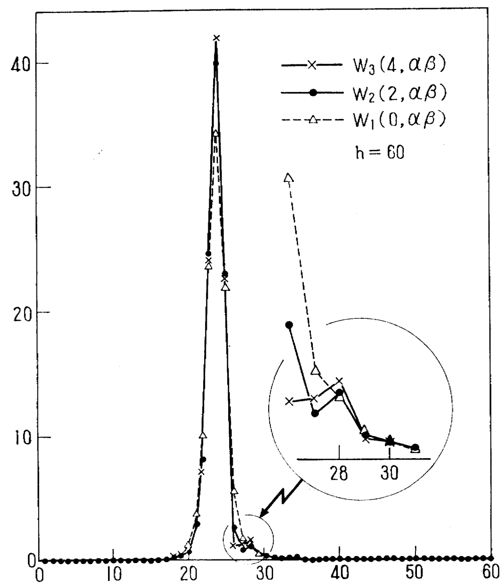


Fig. 2. Power spectrum of the response of a ship model.

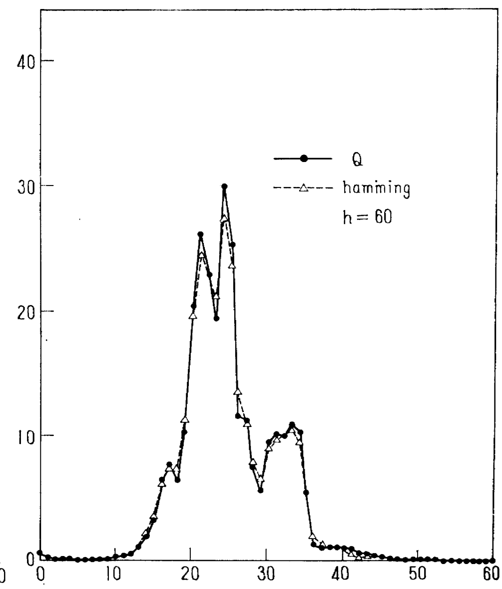


Fig. 3. Power spectrum of the input wave.

When the original data are given in the form $\{x(n\Delta\tau);\ n=1, 2, \cdots, M\}$ for some $\Delta\tau>0$ one can obtain an estimate of the power spectral density function of the process $\{x(n\Delta\tau)\}$ simply by replacing the integrations by the corresponding sums and also $2T$ by $M$. The results of the argument in the preceding sections are all valid for this case, too, and we shall not repeat it here. We have treated the case of continuous time parameter in this paper anticipating the frequent use of the analog equipment for the computation of $C(\tau)$.

Finally we shall illustrate some numerical examples of application of our window to practical problems*. In Fig. 2 there is given a result of comparison of the estimates, obtained by applying $W_1(0, \alpha\beta)$, $W_2(2, \alpha\beta)$ and $W_3(4, \alpha\beta)$ respectively, of the power spectrum of the response of a ship model to an artificially generated random wave. The result shows fairly clearly the effect of the correction for local bias and such alternative applications of the windows of various types will give more insight into the true shape of the spectrum. For this type of application the compatibility, above-mentioned, of the lag windows of trigonometric sum type is quite useful. In Fig. 3 are illustrated the estimates, obtained by

* The data treated here are afforded to the author by courtesy of Dr. Y. Yamanouchi.

applying hamming and $Q$, of the power spectrum of the input wave to the ship model. In this example, too, we can clearly see the effect of the correction for the bias. These corrections are sometimes small in magnitude but may still contain important information, and taking into account that the leakage of the power through the minor side lobes are smaller for $Q$ than for hamming the use of $Q$ in place of hamming will be beneficial when only one type of window is applied.

## Acknowledgment

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