ON EXPONENTIAL AND OTHER RANDOM VARIABLE GENERATORS

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Received Nov. 24, (1961)

Summary

G. Marsaglia [6] proposed a new method for generating exponential random variables. In this note, his method is modified and generalized for generating χ^2 random variables with even degrees of freedom. Remarks refer to general χ^2 and normal random variable generators.

1. Introduction

The procedure to generate an exponential random variable from a uniform random variable is not complicated, it is only to calculate the logarithmic function once. It is, however, interesting to consider alternative approaches based on stochastic models including an exponential distribution. Von Neumann's well-known method [7] is too artificial, and it requires, indeed, more machine time than the straightforward one. Butler's modification [3] seems not an essential improvement. On the other hand G. Marsaglia's method [6] is practical enough.

Let M be a random variable which has the geometric distribution,

$$\Pr(M=m)=(e-1)/e^{m+1}\ , \qquad m=0, 1, 2, \cdots, \tag{1}$$

and let N be one with the 0-truncated Poisson distribution,

$$\Pr(N=n)=1/(e-1)n!\ , \qquad n=1, 2, 3, \cdots. \tag{2}$$

Suppose a sample of random size N, $(U_1, U_2, \cdots, U_N)$, is observed from the uniform distribution on $(0, 1)$. Then a random variable

$$X=M+\min(U_1, U_2, \cdots, U_N) \tag{3}$$

has the exponential distribution

$$\Pr(X<x)=1-e^{-x}\ , \qquad 0<x<\infty\ . \tag{4}$$

It should be noticed that in this method there is freedom in choosing parameters of geometric and truncated Poisson distributions. Their values may be determined properly according to properties of the computer and the uniform random variable generator to be used.

Before discussing this point, a 'statistical meaning' of Marsaglia's method is stated.

2. Generating Poisson process

In Poisson process with parameter λ, the probability that N events occur within one of successive intervals of the same length t is

$$\Pr(N=n)=e^{-\lambda t}(\lambda t)^n/n! , \qquad n=0, 1, 2, \cdots, \tag{5}$$

and under the condition that n events fall in an interval, these events $(V_1, \cdots, V_n)$ has the same conditional distribution as a sample of size n from the $(0, t)$ uniform distribution,

$$n!t^{-n}dv_1dv_2\cdots dv_n , \qquad 0<v_1<v_2<\cdots <v_n<t. \tag{6}$$

Conversely, let $\{N_\nu\}(\nu=1, 2, \cdots)$ be a sequence of independent random numbers with probabilities (5), and let N_ν events fall randomly in the interval $I_\nu=((\nu-1)t, \nu t)$. Denote the coordinates of the events by $Y_1< Y_2<\cdots$. Then the events constitute a Poisson process with parameter λ and $\{X_\nu=Y_\nu-Y_{\nu-1}\}$, $(\nu=1, 2, \cdots; Y_0=0)$, is a sequence of independent random number with the density

$$\lambda e^{-\lambda x} , \qquad 0<x<\infty . \tag{7}$$

This fact is shown, for example, in Doob [4] (VIII § 4). X defined by (3) corresponds to X_1 in case $\lambda=t=1$, the number of vacant intervals preceding the first event is represented by M, and the number of events in the interval where events occur at first is represented by N.

3. Choosing parameters

Now, the new method is described as follows.

Let M be a random variable with the probabilities

$$\Pr(M=m)=(e^\mu-1)/e^{\mu(m+1)} , \qquad m=0, 1, 2, \cdots, \tag{8}$$

where $\mu>0$, and let N be the one with

$$\Pr(N=n)=\mu^n/(e^\mu-1)n! , \qquad n=1, 2, 3, \cdots . \tag{9}$$

Then

$$X=\mu\{M+\min(U_1, \cdots, U_N)\}$$

has the exponential distribution (4).

In this case

$$E_\mu(M+1)=e^\mu/(e^\mu-1) \, , \quad E_\mu(N)=\mu e^\mu/(e^\mu-1) \, . \tag{10}$$

$E_\mu(M+1)$ is a decreasing function and $E_0(M+1)=\infty$, $E_\infty(M+1)=1$, while $E_\mu(N)$ is an increasing one and $E_0(N)=0$, $E_\infty(N)=\infty$. N is the number of times generating uniform random numbers and selecting the minimum of new random number and the previous minimum. M is the number of times looking up the table of cumulative probabilities of M. The latter work may be easier than the former. Denote by $\rho(<1)$ the ratio of machine time which is necessary for looking up an entry of table to one for generating and discriminating a uniform random variable once. The value of μ which minimizes $\rho E_\mu(M+1)+E_\mu(N)$ is a root of

$$e^\mu-(1+\mu)=\rho \, . \tag{11}$$

Some optimum values of μ are shown in Table 1. For a middle size drum computer ρ is about 0.35, and for another larger computer with random access memory and 'table look up' operation code ρ is less than 0.10.

TABLE 1

Optimum value of

ρ	μ	$E_\mu(M+1)$	$E_\mu(N)$
0.72	1.00	1.58	1.58
0.4	.78	1.85	1.44
0.3	.69	2.01	1.38
0.2	.57	2.30	1.31
0.1	.42	2.92	1.22

Although the gain due to the choice of optimum μ is small, $\mu=1/2$ is recommended to larger binary-system computers.

Another value $\mu=\log 2 \doteqdot 0.693$ ($e^\mu=2$) is also recommended for binary-system computers. Because the geometric distribution for this value

$$\Pr(M=m)=2^{-(m+1)} \, , \qquad m=0, 1, 2, \cdots,$$

is easily realized by the 'shift count' operation of uniform random variables.

If the machine time for generating a uniform random number is rather longer, the successive differences $X_2=Y_2-Y_1$, $X_3=Y_3-Y_2$, $\cdots$ may be used.

4. χ^2 random variables

Monte Carlo computation in queuing problems needs often random variables which have the gamma distribution. A usual generating method is to add k exponential random variables to get a χ^2 random variables with $2k$ degrees of freedom. The above analysis suggests the following procedure.

Let $\{M_l\}$ be a sequence of Poisson random integers with mean μ. Let L be a random integer such that the inequalities

$$M_1+M_2+\cdots+M_{L-1}<k\leqq M_1+M_2+\cdots+M_L \,, \tag{12}$$

hold, and put

$$N=k-(M_1+M_2+\cdots+M_{L-1}) \,, \qquad 1\leqq N\leqq M_L \,. \tag{13}$$

Then the random variable

$$X=\mu[L-1+\{\text{the } N\text{th smallest of } (U_1, U_2\cdots, U_{M_L})\}] \tag{14}$$

has the χ^2 distribution with $2k$ degrees of freedom.

$$E(M_l)=\mu \,,$$
$$E(L)=1+\sum_{l=1}^{\infty}\sum_{m=0}^{k-1} e^{-l\mu}\frac{(l\mu)^m}{m!} \,. \tag{15}$$

In this procedure $E(M_l)+E(L)$, the expected number of necessary uniform random variables for generating one χ^2 variable, may be regarded as a criterion for choosing μ. Table 2 suggests a way how to choose μ for some values of k.

TABLE 2

k	μ	$\mu+E(L)$
2	1	3.503
	1.5	3.342
	2	3.591
5	2	5.000
	2.5	4.999
	3	5.164
10	2.5	7.000
	3	6.833
	3.5	6.857
15	3	8.500
	4	8.250
	5	8.500

When k is large, to avoid the use of sequence $\{M_l\}$, more direct selection of L, M_L, N may be suggested. The event E_L that the inequality (13) is satisfied, has the probability

$$\text{(16)}\qquad \Pr(L=l)=\begin{cases} 1-\sum_{j=0}^{k-1} e^{-\mu}\frac{\mu^j}{j!}\ , & l=1\ ,\\ \sum_{j=0}^{k-1}\left\{e^{-\mu(l-1)}\frac{\{\mu(l-1)\}^j}{j!}-e^{-\mu}\frac{\mu^j}{j!}\right\}\ , & l\geqq 2\ ,\end{cases}$$

and under this condition,

$$\Pr(M_L=m|E_1)=e^{-\mu}\frac{\mu^m}{m!}\Big/\Pr(E_1)\ ,\qquad N=k\ ,$$

$$\text{(17)}\qquad \Pr(N=n|E_l)=e^{-\mu(l-1)}\frac{\{\mu(l-1)\}^{k-n}}{(k-n)!}\sum_{j=n}^{\infty}\frac{e^{-\mu}\mu^j}{j!}\Big/\Pr(E_l)\ ,\qquad l\geqq 2\ ,$$

$$\Pr(M_L=m|N=n)=e^{-\mu}\frac{\mu^m}{m!}\Big/\sum_{j=n}^{\infty}\frac{e^{-\mu}\mu^j}{j!}\ ,\qquad m\geqq n\ .$$

Comparing three uniform random variables with three probability tables, two of which have double entries, we may decrease the number of necessary random numbers.

5. Some remarks

In order to get χ^2 random variables with $2n+1$ degrees of freedom, random variables with 1 degree of freedom, which might be generated by the method mentioned below, are added to ones with $2n$ degrees of freedom. For χ^2 with non-integral degrees of freedom ν, I. Takahashi [8] suggested a rejection technique.

Generate χ^2 random variable T with $2n\,(2n<\nu<2n+2)$ degrees of freedom and uniform random variable U. Accept T if

$$\text{(18)}\qquad U<\left(\frac{p}{\varepsilon e(1-p)}\right)^{\varepsilon}T^{\varepsilon}\exp\left(-\frac{p}{1-p}T\right)\ ,\qquad \varepsilon=\nu-2n\ ,\qquad 0<p<1\ ,$$

and reject it if the inequality is not satisfied to get new T and U. Maximum acceptance probability is attained at $p=\varepsilon/\nu$, and it is at worst about 70%.

As exponential random variables are obtained cheaper, generation of normal random variables should be reconsidered. Box and Muller [1] suggested the use of a 'polar representation' of bivariate non-correlated normal variable. Let U_1 and U_2 have uniform distributions.

$$\text{(19)}\qquad \begin{aligned} X&=\sqrt{-2\log U_2}\cos 2\pi U_2\\ Y&=\sqrt{-2\log U_1}\sin 2\pi U_2\end{aligned}$$

are independent normal deviates. In this expression $-\log U_1$ may be replaced by an exponential random variables, and the pair $(\cos 2\pi U_2, \sin 2\pi U_2)$ by

$$\frac{U_1^2-U_2^2}{U_1^2+U_2^2}, \quad \pm\frac{2U_1U_2}{U_1^2+U_2^2}, \qquad \text{if} \quad 0<U_1^2+U_2^2<1, \tag{20}$$

where $\pm$ is a random sign. For computers with floating point arithmetic operation

$$\begin{aligned} X &= \pm U_1\sqrt{2T/(U_1^2+U_2^2)}, \\ Y &= \pm U_2\sqrt{2T/(U_1^2+U_2^2)}, \\ & U_1^2+U_2^2<1, \end{aligned} \tag{21}$$

where T is an exponential random variable and two $\pm$'s are independent random signs, may be preferable. This technique follows from ideas in [5] and [7].

Normal random numbers are generated also from exponential ones making use of a rejection technique [2]. It is based on the factorization

$$\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} = \begin{cases} \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}, & 0<x<1, \\ \sqrt{\frac{1}{2\pi}}e^{-\frac{1}{2}(x-2)^2}2e^{-2(x-1)}, & 1<x<\infty. \end{cases} \tag{22}$$

We generate uniform random variables U and exponential ones T. Choose Procedure I and II respectively with probability 2/3 and 1/3.

$$\begin{aligned} &\text{Procedure I. Accept } U, \text{ if } U^2<2T \\ &\text{Procedure II. Accept } T_1, \text{ if } (1+T_1/2)^2<2T_2 \end{aligned} \tag{23}$$

The average number of operations are shown below. The notation exp. means the generation of an exponential random variable, and similarly unif. The latter method is supposed to be preferable for many computers.

operation	exp.	unif.	sq. root	mult.	div
(19, 20)	1.	2.55	1	3.55	2
(23)	1.60	1.80		1.20	

Finally, a χ^2 random variable with 1 degree of freedom is generated by $T[1+\cos\pi U]$. For $X=\cos\pi U$ a rejection technique due to Butler

[3] is recommended. That is, accept

$$X=1-U_1^2 \quad \text{if} \quad (1+X)U_2^2<1$$

and attach a random sign to X.

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