CONVERGENCE TO BIVARIATE LIMITING EXTREME VALUE DISTRIBUTIONS*)

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1. Introduction

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a sequence of independent bivariate random variables with the common bivariate distribution function (d.f.) $F(x, y)$, and with marginal d.f.'s $F_1(x)$ and $F_2(y)$; let

$$U_n = \max(X_1, \cdots, X_n);\ V_n = \max(Y_1, \cdots, Y_n)\ .$$

The forms of the univariate limiting d.f.'s of U_n and the necessary and sufficient conditions on F_1 for convergence of the d.f. of U_n to one of the limiting forms are well known [2].

It is the object of this paper to establish the conditions under which the random pair (U_n, V_n) has a limiting bivariate distribution. The possible forms of these distributions have been completely discussed in [1], [6], and [7].

In the following it is assumed that the marginal d.f.'s $F_1(x)$ and $F_2(y)$ are such that U_n and V_n each have univariate limiting d.f.'s $\Phi_1(x)$ and $\Phi_2(y)$. This is equivalent to the assertion [2] that there exist sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ such that for all x and y,

$$(1)\qquad \lim_{n\to\infty} F_1^n(a_n x + b_n) = \Phi_1(x)$$
$$\lim_{n\to\infty} F_2^n(c_n y + d_n) = \Phi_2(y)\ .$$

The joint d.f. of (U_n, V_n) is

$$P\{U_n \leqq x,\ V_n \leqq y\} = F^n(x, y);$$

therefore, (U_n, V_n) has a limiting d.f. $\Phi(x, y)$ with marginal limiting d.f.'s $\Phi_1(x)$ and $\Phi_2(y)$ if and only if

$$(2)\qquad \lim_{n\to\infty} F^n(a_n x + b_n,\ c_n y + d_n) = \Phi(x, y)\ .$$

It is shown in [6] that $\Phi(x, y)$ is necessarily of the form

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(3) $$\Phi(x, y)=\Phi_2(y)\Phi_1(x)^{\chi(\log\Phi_2(y)/\log\Phi_1(x))+1} ,$$

where $\chi(t)$ is defined for $t\geqq 0$, is continuous and convex, and satisfies the inequalities

$$\max(-t, -1)\leqq\chi(t)\leqq 0 .$$

2. Conditions for convergence

In the following, it is assumed that $F_1(x)$ and $F_2(y)$ are strictly increasing and continuous, so that they have inverse functions. This assumption is not essential but serves to simplify the proof of the theorem; some sort of "inverse" function can always be constructed for a d.f. (cf. [1]).

Since F_i, $i=1, 2$, have inverses, it is possible to express $F(x, y)$ in the form

$$F(x, y)=H(-\log F_1(x), -\log F_2(y)) ,$$

where $H(u, v)\to 1$ as $(u, v)\to(0, 0)$.

THEOREM 1. *A necessary and sufficient condition that* (U_n, V_n) *have a limiting d.f.* $\Phi(x, y)$ *of the form* (3) *is that for every* $u>0$, $v>0$,

(4) $$\lim_{n\to\infty} n[H(u/n, v/n)-1]=-u[\chi(v/u)+1]-v .$$

PROOF. It will be shown that (2) holds if and only if (4) holds.

(i) *(4)⇒(2)*:

It will be shown that

$$\lim_{n\to\infty} H^n(-\log F_1(a_n x+b_n), -\log F_2(c_n y+d_n))=\Phi(x, y) .$$

By taking logs on both sides of the above relation and using the logarithmic expansion

$$\log H\sim H-1 \qquad (H\to 1) ,$$

one can see that the first assertion is equivalent to

(5) $$\lim_{n\to\infty} n[H(-n^{-1}\log F_1^n(a_n x+b_n), -n^{-1}\log F_2^n(c_n y+d_n))-1]$$
$$= \log\Phi_1(x)[\chi(\log\Phi_2(y)/\log\Phi_1(x))+1]+ \log\Phi_2(y) .$$

If (4) holds for $u>0$, $v>0$, then it holds uniformly in u and v, since

$H(u, v)$ is a monotonic function in each of its variables and χ is a continuous function; then (5) follows immediately from (4).

(ii) (2)$\Rightarrow$(4):

The reasoning in the preceding paragraph shows that it is sufficient to show that (5) implies (4).

Let u and v be any fixed positive numbers. Since $-\log \Phi_1(x)$ and $-\log \Phi_2(y)$ are continuous functions of x and are monotonically increasing [2], there exist numbers w and z such that

$$-\log \Phi_1(w)=u;\ -\log \Phi_2(z)=v\ .$$

It is a consequence of (1) and the monotonicity of the functions Φ_i that for every $\varepsilon>0$, there exists an integer N sufficiently large so that for all $n \geqq N$,

$$\begin{aligned}&-n \log F_1(a_n(w+\varepsilon)+b_n)<u\\&\qquad<-n \log F_1(a_n(w-\varepsilon)+b_n)\ ;\\&-n \log F_2(c_n(z+\varepsilon)+d_n)<v\\&\qquad<-n \log F_2(c_n(z-\varepsilon)+d_n)\ .\end{aligned}$$

Since $H(u, v)$ is monotonically non-increasing in each of its variables,

$$\begin{aligned}&n[H(-n^{-1} \log F_1^n(a_n(w+\varepsilon)+b_n),\ -n^{-1} \log F_2^n(c_n(z+\varepsilon)+d_n))-1]\\&\quad\leqq n[H(u/n,\ v/n)-1]\\&\quad\leqq n[H(-n^{-1} \log F_1^n(a_n(w-\varepsilon)+b_n),\ -n^{-1} \log F_2^n(c_n(z-\varepsilon)+d_n))-1];\end{aligned}$$

as $n \to \infty$, the above inequalities become, by virtue of (5),

$$\begin{aligned}&\log \Phi_1(w+\varepsilon)[\chi(\log \Phi_2(z+\varepsilon)/\log \Phi_1(w+\varepsilon))+1]\\&\qquad+\log \Phi_2(z+\varepsilon) \leqq \varliminf_{n\to\infty} n[H(u/n,\ v/n)-1]\\&\quad\leqq \varlimsup_{n\to\infty} n[H(u/n,\ v/n)-1]\\&\quad\leqq \log \Phi_1(w-\varepsilon)[\chi(\log \Phi_2(z-\varepsilon)/\log \Phi_1(w-\varepsilon))+1]\\&\qquad+\log \Phi_2(z-\varepsilon)\ .\end{aligned}$$

Since ε is arbitrarily small and the extreme terms in the above inequalities are continuous functions of ε, (4) follows.

COROLLARY 1. *U_n and V_n are asymptotically independent if and only if for every $u>0$, $v>0$,*

$$(6)\qquad \lim_{n\to\infty} n(H(u/n,\ v/n)-1)=-u-v\ .$$

PROOF. This follows from the fact that

$$\Phi(x, y)=\Phi_1(x)\Phi_2(y)$$

if and only if $\chi\equiv 0$.

Remark. In applications it is not necessary to compute the function H for all values of u and v but only for those values near (0, 0).

3. Examples

The important case of the bivariate normal distribution has already been treated in [1] and [6] where it was shown that U_n and V_n are asymptotically independent. Other examples will be given here.

(a) The following bivariate distribution may be found in [3]:

$$F(x, y)= \exp\{-[(-\log F_1(x))^m+(-\log F_2(y))^m]^{1/m}\} ; \qquad (m\geqq 1)$$

here,

$$H(u, v)= \exp\{-[u^m+v^m]^{1/m}\} ,$$

and it follows from Theorem 1 that

$$\Phi(x, y)= \exp\{-[(-\log \Phi_1(x))^m+(-\log \Phi_2(y))^m]^{1/m}\} .$$

(b) The following distribution is a generalization of the one considered in [4]:

$$F(x, y)=[(F_1(x))^{-1}+(F_2(y))^{-1}-1]^{-1} ;$$

here,

$$H(u, v)=[e^{-u}+e^{-v}-1]^{-1}$$

and it follows from Theorem 1 that

$$\Phi(x, y)=\Phi_1(x)\Phi_2(y) .$$

4. The k-dimensional case

Let $(X_{1,n}, \cdots, X_{k,n})$ $n=1, 2, \cdots$ be a sequence of independent random k-dimensional vectors with the common multivariate d.f. $F(x_1, \cdots, x_k)$, and marginal d.f.'s $F_i(x)$, $i=1, \cdots, k$. For each n, let

$$Z_{i,n}= \max_{j\leq n} X_{i,j} i=1, \cdots, k .$$

The general form of the k-dimensional limiting d.f. of

$$\bar{Z}_n=(Z_{1,n}, \cdots, Z_{k,n})$$

is unknown. In this section conditions will be given which are sufficient for the convergence of the d.f. of $\bar{Z}_n$ to the product d.f.

$$\Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_k(x_k) \tag{7}$$

where Φ_i, $i=1, \cdots, k$, is a univariate limiting d.f. of $Z_{i,n}$. In this case $Z_{i,n}$, $i=1, \cdots, k$ are asymptotically independent.

Let u_i, $i=1, \cdots, k$, be the least upper bound of all x such that $F_i(x)<1$; u_i may be infinite.

THEOREM 2.*) *Let $F_{i,j}(x_i, x_j)$ denote the bivariate d.f. of $(X_{i,n}, X_{j,n})$. If for every i and j,*

$$\lim_{(x_i,x_j)\to(u_i-,u_j-)} \frac{1-F_i(x_i)-F_j(x_j)+F_{ij}(x_i, x_j)}{1-F_{ij}(x_i, x_j)}=0 \tag{8}$$

then $\bar{Z}_n$ has the limiting d.f. given by (7).

PROOF. Since it has been assumed in the introduction that $F_i(x)$ is in the domain of attraction of $\Phi_i(x)$, (1) holds for k pairs of sequences

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*) Concerning Theorem 2, the referee suggested a necessary condition for asymptotic independence:

Denote

$$\Pr(Z_i>z_i)=\Phi_i$$
$$\Pr(Z_i>z_i,\cdots,Z_k>z_k)=\Phi$$

Then, if

$$\lim_{z_i\to\mu_i} \frac{\Phi}{\max(\Phi_1,\cdots,\Phi_k)}=0 \tag{A}$$

$\bar{Z}_n$ has the limiting d.f. (7).

Because, (10) is expressed as

$$\lim\frac{1-F}{\Sigma\Phi_i}=1$$

and from the relations

$$\Sigma\Phi_i+F-1\geqq\Phi$$
$$\Sigma\Phi_i\leqq k\max(\Phi_1,\cdots,\Phi_k) \tag{B}$$

we have

$$1-\frac{\Phi}{k\max(\Phi_1,\cdots,\Phi_k)}\leqq\frac{1-F}{\Sigma\Phi_i}\leqq 1\,.$$

In case $k=2$, the condition (A), which is slightly weaker than that in Theorem 2, is also sufficient. In case $k\geqq3$, however, it seems that (A) is not sufficient nor the condition in Theorem 2 is necessary.

$\{a_{n,i}\}$ and $\{b_{n,i}\}$, $i=1, \cdots, k$. It will be shown that

$$\lim_{n\to\infty} \frac{F^n(a_{n,1}x_1+b_{n,1}, \cdots, a_{n,k}x_k+b_{n,k})}{F_1^n(a_{n,1}x_1+b_{n,1})\cdots F_k^n(a_{n,k}x_k+b_{n,k})}=1 , \tag{9}$$

which will complete the proof.

After taking logs in (9) and using the logarithmic expansion, it is not hard for one to see that (9) is equivalent to

$$\lim_{n\to\infty} \frac{1-F(a_{n,1}x_1+b_{n,1}, \cdots, a_{n,k}x_k+b_{n,k})}{\sum_{i=1}^{k}[1-F_i(a_{n,i}x_i+b_{n,i})]}=1 . \tag{10}$$

Let $A_i, i=1, \cdots, k$, denote the event

$$\{X_{i,n}>a_{n,i}x_i+b_{n,i}\} ;$$

then (10) is equivalent to

$$\lim_{n\to\infty} P\Big(\bigcup_{i=1}^{k} A_i\Big)\Big/\sum_{i=1}^{k} P(A_i)=1 . \tag{11}$$

Let B_i, $i=1, \cdots, k$, denote the event

$$\{X_{i,n}>x_i\} ;$$

then (8) is equivalent to

$$\lim_{(x_i,x_j)\to(u_i-,u_j-)} \frac{P(B_iB_j)}{P(B_i \cup B_j)}=0 . \tag{12}$$

Now $P(\bigcup_{i=1}^{k} A_i)$ may be written as

$$P\Big(\bigcup_{i=1}^{k} A_i\Big)=\sum_{i=1}^{k} P(A_i)-\sum_{i\neq j} P(A_iA_j)+\cdots ,$$

so that it remains to be shown, from (11), that

$$\lim_{n\to\infty}\left\{1-\frac{\sum_{i\neq j} P(A_iA_j)-\cdots}{\sum_{i=1}^{k} P(A_i)}\right\}=1 . \tag{13}$$

Since, from (12),

$$\frac{P(B_iB_j)}{P(B_i)+P(B_j)}\leqq\frac{P(B_iB_j)}{P(B_i \cup B_j)}\to 0 ,$$

it follows that

$$\frac{P(A_iA_j)}{P(A_i)+P(A_j)}\to 0 ;$$

since there are a finite number of terms in the numerator of the fraction in the brackets in (13), and each is no greater than

$$\max_{i,j} P(A_i A_j) ,$$

the assertion (13) follows and the proof of the theorem is complete.

Remark. It has been shown in [1] that (8) holds for every bivariate normal d.f., so that the theorem holds for every multivariate normal d.f..

5. The case of the minima

All of the results given above for the maxima are analogous to those which are obtainable for the minima; the "isomorphism" between the two cases is discussed in [1] and [5].

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