# ON SOME SEQUENTIAL LIFE TESTS

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## 1. Introduction

B. Epstein and M. Sobel proposed sequential life tests in the exponential case [1], in which a statistic $V(t)$, called "total life", is observed continuously in time $t$ and decision is made at an instant when $V(t)$ does cross the preassigned limits. In the nonreplacement case of that test, the probability that a decision can be made after all $n$ items on test have failed is smaller than one. Therefore the average test time can not remain to be finite so long as $n$, the number of test items simultaneously placed on test, is finite. Thus they recommended the sequential test, in which $n$ must be determined so large that the probability not to reach any decision is negligible small and some decision rules must be defined in advance to provide for such indeterminable cases. However, it seems to be expensive and troublesome that a sufficiently large number of items are placed on test simultaneously, because test equipments must be so large according to the number of test items.

In this paper, the test procedures applicable to the wider class of distributions including the exponential case will be proposed, in which a suitable number of test items are placed on test repeatedly untill it reaches a decision. The operating characteristic function, the average failure (sample) number and the average test time will be obtained in comparison with the usual sequential test without a continuous time parameter.

## 2. Formulation of the problems and two lemmas

Suppose that the underlying probability density function for length of life is $f(x, \theta)$. The problem is to test a simple hypothesis $H_0 : \theta=\theta_0$ against a simple alternative $H_1 : \theta=\theta_1 (\theta_1<\theta_0)$, with test time as short as possible and with errors of the first and second kinds equal to the preassigned values $\alpha$ and $\beta$ respectively. The test procedure in the nonreplacement case is defined as follows:

a) The first sample of size $n_1$ is taken from the population and placed on test simultaneously. Failure items are not replaced.

b) The probability ratio

$$\frac{P_1(x_{11}, \cdots, x_{1k_1}, t; \theta_1)}{P_1(x_{11}, \cdots, x_{1k_1}, t; \theta_0)} = \prod_{i=1}^{k_1} \frac{f(x_{1i}, \theta_1)}{f(x_{1i}, \theta_0)} \cdot \left[\frac{1-F(t, \theta_1)}{1-F(t, \theta_0)}\right]^{n_1-k_1} \tag{1}$$

is observed continuously in time $t$, where $k_1$ and $x_i$ denote the number of failures and the time of the $i^{\text{th}}$ failure up to time $t$ respectively, and $F(x, \theta)$ is the distribution function, and it is compared continuously with constants $A$ and $B$ corresponding to $\alpha$ and $\beta(0<B<1<A)$.

c) So long as the probability ratio (1) remains within the limits $A$ and $B$, namely, the inequalities

$$B < \frac{P_1(x_{11}, \cdots, x_{1k_1}, t; \theta_1)}{P_1(x_{11}, \cdots, x_{1k_1}, t; \theta_0)} < A \tag{2}$$

hold, the test is continued; as soon as the equality

$$B = \frac{P_1(x_{11}, \cdots, x_{1m_1}, T_1; \theta_1)}{P_1(x_{11}, \cdots, x_{1m_1}, T_1; \theta_0)} \tag{3}$$

holds for some integer $m_1$ and time $T_1$, the test is stopped and $H_0$ is accepted; as soon as the inequality

$$\frac{P_1(x_{11}, \cdots, x_{1m_1}; \theta_1)}{P_1(x_{11}, \cdots, x_{1m_1}. \theta_0)} \geqq A \tag{4}$$

holds for some integer $m$ and $t = x_{m_1}$, the test is stopped and $H_0$ is rejected.

d) If it happens that any decision can not be made after all items in the first sample have failed, then the second sample of $n_2$ items is taken and placed on test.

e) The probability ratio

$$\frac{P_1(x_{11}, \cdots, x_{1n_1}; \theta_1)}{P_1(x_{11}, \cdots, x_{1n_1}; \theta_0)} \cdot \frac{P_2(x_{21}, \cdots, x_{2k_2}, t; \theta_1)}{P_2(x_{22}, \cdots, x_{2k_2}, t; \theta_0)} \tag{5}$$

is observed continuously in time $t$, where the first and second terms correspond to the probability ratios for the first and second samples and $x_{2i}$ and $t$ are measured from the beginning of the test for the second sample.

f) The decision rule for the ratio (5) is the same as stated in $c$.

$g$) If any decision can not be made after all $n_2$ items in the second sample have failed, then the third sample of $n_3$ items is taken and placed on test. And so on $\cdots\cdot$.

The test procedure defined above has a mixed character of continuous and discrete tests such that the observation is made continuously in time $t$ and the sample is replaced by a new one when all items in the sample have failed. Therefore, it is clear that the average failure (sample) number is finite and the average test time remains to be finite as in the case of discrete (item by item) tests defined by A. Wald. In these points, this test is improved compared with the test defined by B. Epstein and M. Sobel. Moreover, $n_i$'s may be chosen so that the cost for test is reduced as much as possible, and the optimal $n(=n_i)$ may exist if the loss function is chosen suitably for the test—$n_i$'s may be different from a sample to another, but it is convenient from a practical point of view that they are chosen all the same.

**Remark.** In observing probability ratio (1), it is usually monotone decreasing and continuous in $t(x_{i-1}<t<x_i)$, because the ratio $(1-F(t,\theta_1))/(1-F(t,\theta_0))$ has the same property in many cases under the condition $\theta_0>\theta_1$. Moreover, the probability ratio (1) is changed by factor

$$\frac{f(x_i,\theta_1)}{1-F(x_i,\theta_1)}\Big/\frac{f(x_i,\theta_0)}{1-F(x_i,\theta_0)}$$

when the $i^{\text{th}}$ failure occurs at $t=x_i$, where $f(x,\theta)/(1-F(x,\theta))$ represents the instantaneous failure rate at $t=x$ and usually monotone decreasing in $\theta$ (scale or location parameter). Therefore, it seems to be valid in many cases that the probability ratio (1) increases every time when failures occur and decreases during the time between failures.

Consequently, the equality (3) holds when accepting $H_0$, and the inequalities (4) hold when rejecting $H_0$.

Finally, two lemmas are stated for the following sections.

LEMMA 1. *Let $x$ and $y$ be two random variables with probability density functions $f(x,\theta)$ and $g(y,\theta)$, $\theta$ lying in some parameter space $\Omega$. If the random variable $y$ is a sufficient statistic of $x$ for the parameter $\theta$, then the sequential probability ratio tests for $x$ and $y$, excluding time parameter $t$ and testing the simple hypothesis $\theta=\theta_0$ against the simple alternative $\theta=\theta_1$, are equivalent.*

PROOF. There exists a transformation $y=\varphi(x)$ from $x$ to $y$ and a function $h(x)$ independent of $\theta$ for which the relation

$$(6)\qquad f(x, \theta)=h(x)g(\varphi(x), \theta)$$

holds, because the random variable $y$ is sufficient for $\theta$ by assumption. Therefore, the equality

$$(7)\qquad \frac{f(x, \theta_1)}{f(x, \theta_0)}=\frac{g(y, \theta_1)}{g(y, \theta_0)}, \qquad y=\varphi(x),$$

holds for all $\theta_0$ and $\theta_1$ in $\Omega$. Consequently, the probability ratios

$$\prod_{i=1}^{k}\frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}, \qquad \prod_{i=1}^{k}\frac{g(y_i, \theta_1)}{g(y_i, \theta_0)}, \qquad (k=1, 2, \cdots)$$

have the same distribution functions, and the probability ratio tests for $x$ and $y$ are equivalent in the sense that the operating characteristic functions are identical and the average sample numbers are equal, while the average test time may be different.

COROLLARY. *If there exists a functional relation $y=\varphi(x)$ independent of the parameter $\theta$ between random variables $x$ and $y$, which is monotone and differentiable in the whole region of $x$, then the sequential probability ratio tests for $x$ and $y$ are equivalent.*

PROOF. As the relation $f(x, \theta)=\varphi'(x)g(\varphi(x), \theta)$ holds by assumption, $y$ is sufficient for $\theta$, so the condition of the lemma 1 is satisfied.

**Remark.** Clearly $E_\theta(x)=E_\theta(y)$ does not necessarily holds, so the probability ratio tests for $x$ and $y$ are not equivalent with respect to the average test time.

LEMMA 2. *Under the conditions of lemma 1 and the condition that the function $y=\varphi(x)$ is monotone increasing and differentiable in $x$, the probability ratio tests for $x$ and $y$ including continuous time parameter $t$ in the nonreplacement case, are equivalent excepting the average test time.*

PROOF. In the probability ratio given in (5), it is clear that the first terms for $x$ and $y$ are identical by Lemma 1, so it is sufficient to show that the second term for $x$ and $y$ in (5) are identical. Namely, it is sufficient to show that the relation

$$(8)\qquad \prod_{i=1}^{m}\frac{f(x_i,\theta_1)}{f(x_i,\theta_0)}\cdot\left[\frac{1-F(t,\theta_1)}{1-F(t,\theta_0)}\right]^{n-m}=\prod_{i=1}^{m}\frac{g(y_i,\theta_1)}{g(y_i,\theta_0)}\cdot\left[\frac{1-G(u,\theta_1)}{1-G(u,\theta_0)}\right]^{n-m}$$

holds, where $y_i=\varphi(x_i)$ and $u=\varphi(t)$. Since the first terms of both members in (8) are identical by lemma 1, it is sufficient to show that $F(t,\theta)=G(u,\theta)$ for all $\theta$ in $\Omega$. By assumption for the function $y=\varphi(x)$, monotone increasing and differentiable in $x$, the relation

$$(9)\qquad f(x,\theta)=\varphi'(x)g(\varphi(x),\theta)$$

holds for all $\theta$. Consequently, $F(x,\theta)=G(u,\theta)$ holds for all $\theta$ by integrating both members of (9) from 0 to $t$, considering that the interval $(0,t)$ in $x$ corresponds to $(0,u)$ in $y$.

## 3. The sequential tests in the exponential case

### 3.1 Case of $n_i=1$

Suppose that $k-1$ failures have been observed up to the present time and $k^{\text{th}}$ item is now on test, and let $x_i$ denote the life time of the $i^{\text{th}}$ item. Then the probability ratio to obtain such observations is

$$(10)\qquad \left(\frac{\theta_0}{\theta_1}\right)^{k-1}\exp\left\{-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left(\sum_{i=1}^{k-1}x_i+t\right)\right\}$$

where $t$ denotes the time from the $(k-1)^{\text{st}}$ failure to the present time. So long as the probability ratio (10) remains within limits $A$ and $B$, that is, the inequalities

$$(11)\qquad \log B<(k-1)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left(\sum_{i=1}^{k-1}x_i+t\right)<\log A$$

hold, the test is continued; the test is stopped and $H_0$ is accepted as soon as the equality

$$(12)\qquad \log B=(m-1)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left(\sum_{i=1}^{m-1}x_i+T\right)$$

holds for some integer $m$ and some $T(0<T<x_m)$; the test is stopped and $H_0$ is rejected as soon as the inequality

$$(13)\qquad m\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^{m}x_i\geqq\log A$$

holds for some integer $m$(see Fig. 1).

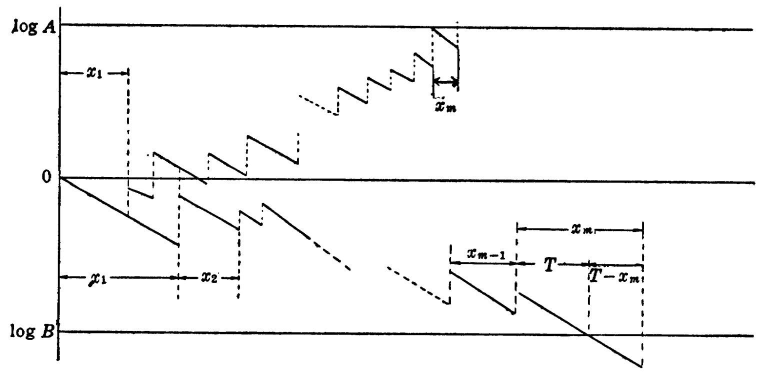


Fig. 1

The test procedure stated above, including continuous time parameter $t$, will be called "test I", and the equivalent test procedure stated below, excluding parameter $t$, be called "test II". The procedure of test II is as follows:

a) The same boundaries $A$ and $B$ are used as in the case of test I.

b) Observation is not made continuously in time, but every time when failures occur.

c) The test is continued so long as the inequalities

$$\log B+\log\frac{\theta_0}{\theta_1}<\sum_{i=1}^{k} z_i<\log A \tag{14}$$

holds; the test is stopped and $H_0$ is accepted as soon as the inequality

$$\log B+\log\frac{\theta_0}{\theta_1}\geqq\sum_{i=1}^{m} z_i \tag{15}$$

holds for some integer $m$; the test is stopped and $H_0$ is rejected as soon as the inequality

$$\sum_{i=1}^{m} z_i\geqq\log A \tag{16}$$

holds for some integer $m$, where

$$z_i=\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)x_i\ .$$

It is easily seen from Fig. 1 and the definitions of test I and

II that both acceptance and rejection of $H_0$ are equivalent for tests I and II excepting the test time and number of failures. Then the operating characteristic functions of test I and II are identical to $L(\theta)$, say, and errors of the first and second kinds for tests I and II are also identical to $\alpha$ and $\beta$, respectively. Let $r_1$ and $r_2$ denote the failure numbers for tests I and II, respectively. Then the following relation

$$\begin{aligned} r_1 &= r_2 - 1 \qquad \text{if } H_0 \text{ is accept,} \\ &= r_2 \qquad\quad\ \text{if } H_0 \text{ is rejected,} \end{aligned} \tag{17}$$

holds. Taking expectations for both sides of (17), we obtain

$$E_\theta(r_1) = L(\theta) \cdot E_\theta(r_2 - 1) + (1 - L(\theta)) \cdot E_\theta(r_2) \tag{18}$$

or

$$E_\theta(r_1) = E_\theta(r_2) - L(\theta) \,.$$

In particular,

$$\begin{aligned} E_{\theta_0}(r_1) &= E_{\theta_0}(r_2) - (1-\alpha) \\ E_{\theta_1}(r_1) &= E_{\theta_1}(r_2) - \beta \end{aligned} \tag{19}$$

namely, the average failure number in test I is smaller than that in test II by $L(\theta)(\leqq 1)$. In this respect, test I is a little improved test procedure compared with test II, which is identical to a usual sequential probability ratio test having the boundaries $A$ and $B \cdot (\theta_0/\theta_1)$. This is easily seen from (14)~(16). Consequently, the operating characteristic functions and the average failure numbers in tests I and II are approximated by the inequalities (20) and (21)

$$\frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}} \leqq L(\theta)(\text{or } 1 - L(\theta)) \leqq \frac{\delta_\theta A^{h(\theta)} - 1}{\delta_\theta A^{h(\theta)} - B^{h(\theta)}} \,, \quad \text{if } h(\theta) > 0 (< 0) \,, \tag{20}$$

where

$$\theta = \frac{\left(\frac{\theta_0}{\theta_1}\right)^{h(\theta)} - 1}{h(\theta)\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} \quad \text{and} \quad \delta_\theta = \frac{1 - \exp\left(-\frac{a}{a-1}\log a\right)}{1 - \exp\left(-\frac{1}{a-1}\log a\right)} \,.$$

$$\frac{1}{E_\theta(z)}\{L(\theta)\log B + (1 - L(\theta))\log A\} \underset{(\geqq)}{\leqq} E_\theta(r_1) \underset{(\geqq)}{\leqq} \frac{1}{E_\theta(z)}\{L(\theta)\log B + (1 - L(\theta))(\log A + \xi_\theta)\} \,, \tag{21}$$

if $\quad E_\theta(z) \underset{(<)}{>} 0$ , where $E_\theta(z) = \log\frac{\theta_0}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\theta$ and

$$\xi_\theta = b\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\left(1 - \exp\left(-\frac{b}{\theta}\right)\right)^{-1} - \theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right), b = \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^{-1}\log\frac{\theta_0}{\theta_1} .$$

These inequalities are obtained in the same way given by A. Wald (see [3]).

Regarding the average sample number for test I, including all items on test which does not necessarily fail, it is cleary identical to the average failure number $E_\theta(r_2)$ for test II. Finally, the average test time will be obtained. It is easily seen that the formulas (22) hold, supposing that the underlying distribution is exponential;

$$(22) \qquad \begin{aligned} E_\theta(x_1 + \cdots + x_{r_1} + T) &= \theta \cdot E_\theta(r_1) \\ E_\theta(x_1 + \cdots + x_{r_2}) &= \theta \cdot E_\theta(r_2) . \end{aligned}$$

First we prove (22) for test I.

For sufficiently large fixed integer $N$, the sum of random variables $x_1 + x_2 + \cdots + x_N$ is split into two parts $S_{r_1}$ and $S'_{r_1}$ such that $S_{r_1} = x_1 + \cdots + x_{r_1} + T$ and $S'_{r_1} = (x_{r_1+1} - T) + x_{r_1+2} + \cdots + x_N$ if $r_1 < N$. Therefore, the equalities $N\theta = E_\theta(x_1 + x_2 + \cdots + x_N) = P_N E_{\theta N}(S_{r_1} + S'_{r_1}) + (1 - P_N)E^*_{\theta N}(x_1 + \cdots + x_N)$ hold, where $P_N$ denotes the probability that $r_1 < N$ and $E_{\theta N}$ and $E^*_{\theta N}$ denote the conditional expectations, given $r_1 < N$ and $r_1 \geqq N$, respectively. $(1 - P_N)E^*_{\theta N}(x_1 + \cdots + x_N)$ tends to 0 when $N$ tends to infinity as shown by A. Wald. While

$$P_N E_{\theta N}(S'_{r_1}) = \sum_{k=0}^{N-1} p_k E_{\theta k}(S'_k) = \sum_{k=0}^{N-1} p_k(N-k)\theta = P_N N\theta - \theta\sum_{k=0}^{N-1} kp_k ,$$

where $p_k$ denotes the probability that test I terminates with $k$ failures. Consequently,

$$P_N E_{\theta K}(S_{r_1}) = \theta\left\{\sum_{k=0}^{N-1} kp_k + (1 - P_N)(N + E^*_{\theta N}(x_1 + \cdots + x_N))\right\}$$

holds, and by letting $N$ tend to infinity, we obtain

$$E_\theta(S_{r_1}) = \theta \cdot \sum_{k=0}^{\infty} kp_k = \theta E_\theta(r_1) .$$

In the case of test II also, we can prove (22) in the same way.

### 3.2 The replacement case

The results in this section are useful for understanding the next section.

The test procecure is as follows;

a) $n$ items are placed on test at the same time, which are drawn from the population having the probability density function

$$f(x, \theta)=\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \qquad x>0 . \tag{23}$$

b) A new item is placed on test instead of a failure as soon as it occurs.

c) The test is continued as long as the inequalities

$$\log B<(k-1)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left(\sum_{i=1}^{k-1} y_i+u\right)< \log A \tag{24}$$

hold; the test is stopped and $H_0$ is accepted as soon as the equality

$$\log B=(m-1)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left(\sum_{i=1}^{m-1} y_i+U\right) \tag{25}$$

holds for some integer $m$ and some $U(0<U<y_m)$; the test is stopped and $H_0$ is rejected as soon as the inequality

$$m\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^{m} y_i\geqq \log A \tag{26}$$

holds for some integer $m$, where $y_i$ denotes the time from the $(i-1)^{st}$ failure to the $i^{th}$ failure and $u$ denotes the time from the $(k-1)^{st}$ failure to the present time.

The test procedure defined above will be called "test III". Clearly, the random variable $y_i$ has the probability density function

$$g(y, \theta)=\frac{n}{\theta}\exp\left(-\frac{n}{\theta}y\right), \tag{27}$$

because it is the smallest value of the sample of size $n$ drawn from the population having the probability density function (23). Here we consider the transformations $x=ny$ and $t=nu$, which are monotone increasing and differentiable in $y$ and $u$. Then the sequential probability ratio tests for $x$ and $y$ are equivalent excepting the average test time by Lemma 2. Therefore, test III has the same operating characteristic function and average failure (sample) number as test I. As for the average test time, the formula

$$E_\theta(y_1+\cdots+y_{r_1}+U)=\frac{\theta}{n}E_\theta(r_1) \tag{28}$$

holds. Namely, the average test time of test III is reduced to $1/n$ times that of test I.

### 3.3 Nonreplacement case ($n_i=n$)

The test procedures $a\sim g$ are stated in section 2, and the probability ratio (5) is represented as

$$\left(\frac{\theta_0}{\theta_1}\right)^{\sum_{i=1}^{m}n_i+l}\exp\left[-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}x_{ij}+\sum_{j=1}^{l}x_{mj}+(n_m-l)v\right\}\right], \tag{29}$$

where $x_{ij}$ denotes the $j^{\text{th}}$ smallest life time in the $i^{\text{th}}$ sample,

in the case when the $(m+1)^{\text{st}}$ sample are now on test without making any decision after all items in the first $m$ samples have failed and $l$ failures in the $(m+1)^{\text{st}}$ sample have been observed up to the present time. This test will be called "test IV". If numbers of the sample are all identical to $n$, the logarithm of probability ratio (29) is equal to

$$(mn+l)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}+\sum_{j=1}^{l}x_{mj}+(n-l)v\right\}. \tag{30}$$

Now, by the relations

$$\sum_{j=1}^{n}x_{ij}=\sum_{j=1}^{n}(n-j+1)(x_{ij}-x_{i,j-1}) \tag{31}$$
$$\sum_{j=1}^{l}x_{ij}+(n-l)v=\sum_{j=1}^{l}(n-j+1)(x_{mj}-x_{m,j-1})+(n-l)(v-x_{ml})$$

and the transformations $y_{ij}=x_{ij}-x_{i,j-1}$ and $u=v-x_{ml}$, the logarithm of the probability ratio (30) is transformed into

$$(mn+l)\log\frac{\theta_0}{\theta_1}-\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(n-j+1)y_{ij}+\sum_{j=1}^{l}(n-j+1)y_{mj}+(n-l)u\right\}. \tag{32}$$

Since the random variable $y_{ij}$ represents the time between the $(j-1)^{\text{st}}$ and $j^{\text{th}}$ failures in the $i^{\text{th}}$ sample of $n$ items, it has the same form of probability density function (27) replacing $n$ by $n-j+1$. Therefore, test IV is easily proved to be equivalent to test I as in the proof of test III by means of transformations $z_{ij}=(n-j+1)y_{ij}$——$z_{ij}$ has the probability density function (23).

The average test time for test IV is obtained by the formula,

$$E_\theta\Big(\sum_{i=1}^{m} x_{in}+v\Big)=\theta\sum_{h=0}^{\infty}\sum_{l=1}^{n} p_{hl}\Big(h\sum_{j=1}^{n}\frac{1}{n-j+1}+\sum_{j=1}^{l}\frac{1}{n-j+1}\Big) \tag{33}$$

where $p_{hl}$ denotes the probability that test I terminates with $hn+l$ failures. The proof of (33) can be made in the same way as that of the formula (22).

## 4. Applications to the distributions other than the exponential one

### 4.1 Applications to Weibull distribution

The probability density function of the Weibull distribution is given by

$$g(z, \theta)=\frac{1}{\theta}pz^{p-1}\exp\Big(-\frac{z^p}{\theta}\Big), \qquad z>0 . \tag{34}$$

where $p$ is called a shape parameter ($p>1$). In this case, the mean life is given by

$$m_p=\frac{\Gamma(1/p)}{p}\theta^{1/p} . \tag{35}$$

The hypothesis for the mean life $m_p$ is equivalent to the hypothesis for the parameter $\theta$, because the parameter $p$ is a fixed constant which determines the distribution with the parameter $\theta$. So it is sufficient to consider a simple hypothesis $\theta=\theta_0$ and a simple alternative $\theta=\theta_1$ instead of the hypothesis for the mean life $m_p$. From the transformation $y=z^p$, it is seen that the probability density function of $y$ is identical to that of $x$ in (23). Then the probability ratio tests for $y$ and $z$ are equivalent by Lemma 2. Namely, the sequential probability ratio test in Weibull's case, including time parameter has the same operating characteristic function $L(\theta)$ and the same average failure number $E_\theta(r_1)$ as those of test IV equivalent to test 1. The average test time in the Weibull's case is difficult to obtain exactly, because times between failures are not mutually independent and lengths of lives ($z_{i1}, z_{i2}, \cdots, z_{in}$) must be considered to be order statistics in the $i^{\text{th}}$ sample. As to an upper bound for the average test time, we may use the inequality

$$E_\theta\Big(\sum_{i=1}^{m} z_{in}+W\Big)\leqq\mu_n\cdot\sum_{k=1}^{\infty} kP_k , \tag{36}$$

where $\mu_n$ is the expectation of the largest value in a sample of size $n$ from (34) and $P_k$ is the probability that a sequential test in Weibull's case terminates at the $k^{th}$ sample. The value of $\mu_n$ can be calculated numerically by

$$\mu_n = n\cdot\theta^{1/p}\int_0^1\left(\log\frac{1}{1-y}\right)^{1/p} y^{n-1}dy\ . \tag{37}$$

However, the values of $P_m$'s are not known exactly, and it is difficult to evaluate (36). Therefore, the only remaining way is to evaluate it directly by the Monte-Carlo method. The arguments stated above are applicable to the family of distributions having the probability density function

$$h(z,\theta) = \frac{\varphi'(z)}{\theta}\exp\left(-\frac{\varphi(z)}{\theta}\right),\qquad 0\leqq z<\infty\ , \tag{38}$$

where $\varphi(z)$ is monotone increasing and differentiable in $z$ and $\varphi'(z)/\theta$ is called "instantaneous failure rate" or "hazard rate". Namely, the sequential probability ratio test for $z$, including time parameter, is equivalent to test I, excepting the average test time.

### 4.2 Application to the log-normal distribution

Suppose that the underlying distribution has the probability density function

$$g(y,\theta) = \frac{1}{y\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\log y-\theta)^2\right),\qquad y>0\ . \tag{39}$$

Then the probability ratio test for $y$ including time parameter $u$ is equivalent to the test for $x$, where $x$ has the probability density function

$$f(x,\theta) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x-\theta)^2\right),\qquad -\infty<x<\infty\ , \tag{40}$$

by considering the transformation $x=\log y$. Values of ratios (1) are very troublesome to calculate, because the distribution function $F(x,\theta)$ or $G(y,\theta)$ is not expressed explicitly. Therefore, it is not practical to apply the sequential probability ratio test, including continuous time parameter, to $x$ or $y$ unless a high speed computer can be available. Such circumstances will arise in general distributions other than the exponential one, so it may be rather practical to calculate the probability ratio (1) every time failures do occur than to observe (1) continuously in time $t$.

## 5. Conclusion

The sequential probability ratio tests I$\sim$IV proposed in this paper (replacement and nonreplacement cases), are all equivalent for the exponential distributions, excepting the average test times. Namely, they have the same characteristic function $L(\theta)$ and the less average failure number by $L(\theta)$ as compared with usual sequential probability ratio tests defined by A. Wald.

The replacement tests is recommended for the case of exponential distribution from economical and practical points of view, because test equipments are utilized completely and test time can be reduced as much as possible in that case. However the replacement test is not difficult to be applicable to general distributions, because times between failure are not mutually independent. We have to utilize the sequential probability ratio test in the nonreplacement case observing the probability ratio every time failures occur. It is very interesting but difficult to obtain analytically the operating characteristic function and the average failure (sample) number, and it is much more difficult to obtain the average test time, for sequential probability ratio tests including time parameter $t$ in general cases. It seems that we can get such results more easily and successfully by the Monte-Carlo method than by analytical methods.

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