SOME MODELS CONCERNING STATISTICAL TREATMENT OF A CERTAIN CONGESTION PHENOMENON

BY TOSIO UEMATU

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1. Introduction and summary

In this paper we treat a stochastic model of moving particles, in connection with a certain congestion phenomenon which arises when a crowd of people swarm in a narrow area.

The model we consider becomes, under some natural assumptions, a homogeneous Markov chain with a transition pattern, and is similar to the Ehrenfest model which can be interpreted as diffusion with a central force (Feller [1]). However, in our model, more restrictions to the movement of the particles are imposed than in the Ehrenfest model. Moreover, our model is treated two-dimensionally.

In section 2 we shall define this model, and in section 3 we shall derive the limiting distribution of the position of particle when the time tends to infinity. Generally, this limiting distribution is seen to be such that the central part has more probability mass than the peripheral part, which means that we have more congestion in the central part.

From a practical viewpoint, the study of actual occurrence of the situation above the critical level of congestion is more important than that of the limiting behavior. We treat this problem in section 4. However it is very difficult to deal with this problem on the basis of the model considered, so we set up another model which is simpler than the original one. We take up the time of duration above and below the critical level on the basis of this simplified model. We shall give the means and variances of these times.

2. A model concerning a certain congestion phenomenon

The congestion phenomenon we are going to consider here is concerning a time-honored custom, in Japan, called "motimaki" in Japanese. This is a custom of scattering about a lot of rice cakes ("moti" in Japanese) in celebration of a certain happy event. These cakes symbolize good luck, so that people swarm to get the scattered cakes. The open space, where the event takes place, is crowded with people, and cakes are thrown toward the crowd from a scaffold. Every time cakes are thrown the crowd moves toward the falling, giving rise to a congestion of considerable degree.

It is conceivable in this process that there will be many critical moments when the congestion happens to come up to such a level that a mere chance might cause some serious accident or other. The occurrence of the situation above such a critical level may be the natural consequence originating from the mechanism of the movement of the crowd. Our main concern is this point.

In this section we formulate a model which represents the mechanism of the above-mentioned process of "motimaki". Instead of the area in which the process goes on we consider a rectangle A. Let the breadth and length of A be M and M' respectively, both being positive integers. We partition A into MM' small squares with the side of unit length, and consider particles each of which moves about from one square to another. Each of these particles represents the individual member of the crowd gathering in the area to pick up the cakes. Now we represent these squares by co-ordinates (i, j), $i=1, 2, \cdots, M; j=1, 2, \cdots, M'$.

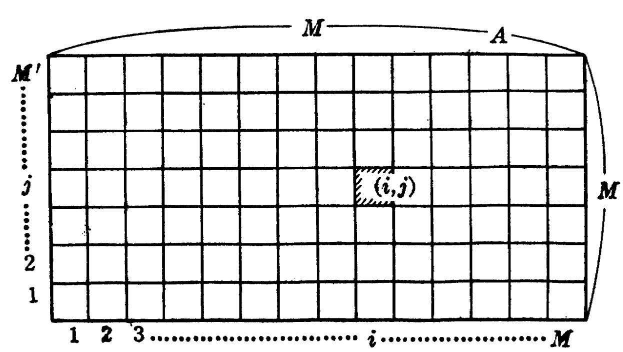

Fig. 1.

We consider that the cakes are thrown on one of the squares at regular time intervals, and that every falling gives an impetus which causes the crowd to move toward the spot of falling. To formulate this we assume that at regular time intervals one of the squares is selected, and that at the same time each particle is stimulated to move to the direction of the selected square.

As for the selection among the squares and the movement of any particle toward the selected square under the attraction we make the following assumptions:

(i) Each square is selected at regular time intervals with equal probability, that is, the cakes are thrown on the area at random. The consecutive selections are independent.

(ii) Every time a square is selected, each of the particles may move into some neighboring square, which is determined by the following rules (iii), (iv) and (v), but it may also occur that some of the particles do not move by some selections according to these rules.

(iii) There are some integers R and $R'(1\leqq R\leqq M-1, 1\leqq R'\leqq M'-1)$ such that only the particles on the square with the co-ordinates (k, l) satisfying the condition

$$i-R\leqq k\leqq i+R\,, \qquad j-R'\leqq l\leqq j+R' \tag{1}$$

may move, when the selected square has the co-ordinates (i, j).

This means that members of the crowd distant from the spot of falling do not move at all.

(iv) When a square (i, j) was selected, any particle in the squares with co-ordinates (k, l) satisfying (1) has a positive probability $\lambda(0<\lambda\leqq 1)$ of moving to a neighboring square, which is determined according to the rule stated in (v) below.

This assumption expresses the situation that in the movement of the crowd there is disturbance which is caused by the mutual jostling among members of the crowd.

Now, for simplicity, we assume that at each falling the movements to the neighboring squares of the particles in the domain of the above-mentioned squares, are mutually independent.

(v) The neighboring square mentioned in (iv) is the one with the co-ordinates (k', l') such that

$$k'=\begin{cases} k-1 & \text{if} \quad i<k\,, \\ k & \text{if} \quad i=k\,, \\ k+1 & \text{if} \quad i>k\,, \end{cases}$$

$$l'=\begin{cases} l-1 & \text{if} \quad j<1\,, \\ l & \text{if} \quad j=k\,, \\ l+1 & \text{if} \quad j>1\,. \end{cases}$$

In the next section we deal with the movement of an arbitrary patricle on A on the assumptions (i), (ii), (iii), (iv) and (v).

3. The limiting distribution of the position of a particle

In this section we derive the limiting distribution of the position of an arbitrary particle in the model stated in the previous section when the time tends to infinity.

By assumption it is clear that the system of the movement of an arbitrary particle forms a homogeneous Markov chain with the states (i, j), $i=1, 2, \cdots, M; j=1, 2, \cdots, M'$. We denote the transition probabilities of the chain by $p_{(ij)(kl)}$, $i, k=1, 2, \cdots, M; j, l=1, 2, \cdots, M'$.

Now for each (i, j), we consider the rectangle $I_{(ij)}$, which is the common part of A and the rectangle with the center (i, j) and having the breadth and length of $2R+1$ and $2R'+1$ respectively. The sides of $I_{(ij)}$ is partitioned into six parts I_1, I_2, I_3, I_4, I_5 and I_6 as shown in Fig. 2. I_2 and I_5 have unit length. We denote the length of I_1, I_3, I_4 and I_6 by $M\alpha_{i,i-1}$, $M\alpha_{i,i+1}$, $M'\beta_{j,j-1}$, $M'\beta_{j,j+1}$ respectively and put $\alpha_{ii}=1/M$, $\beta_{jj}=1/M'$. Moreover we define $\alpha_{ik}=\beta_{jl}=0$ for all integers $k\neq i-1, i, i+1$

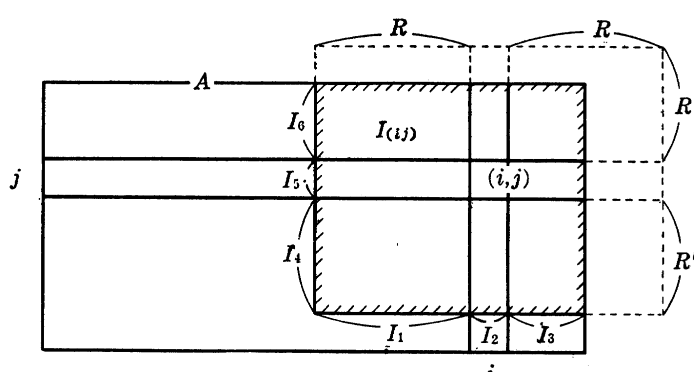

Fig. 2.

and $l\neq j-1, j, j+1$, and put

$$A_i=\alpha_{i,i-1}+\alpha_{ii}+\alpha_{i,i+1}=\sum_{k=1}^{M}\alpha_{ik}\ , \tag{2}$$

$$B_j=\beta_{j,j-1}+\beta_{jj}+\beta_{j,j+1}=\sum_{l=1}^{M'}\beta_{jl}. \tag{3}$$

Then it can easily be seen that the transition probabilities $p_{(ij)(kl)}$ are expressed as follows:

$$p_{(ij)(kl)}=\lambda\alpha_{ik}\beta_{jl} \quad \text{for} \quad i\neq k \quad \text{or} \quad j\neq l \tag{4}$$

and

$$p_{(ij)(ij)}=1+\lambda(\alpha_{ii}\beta_{jj}-A_iB_j). \tag{5}$$

Now it is obvious that the chain is irreducible, and, being finite chain, it is ergodic. That is, the n-step transition probabilities $p^{(n)}_{(ij)(kl)}$ have the limits $\xi_{(kl)}$, which are independent of the initial state (i, j), when n tends to infinity, $i, k=1, 2, \cdots, M; j, l=1, 2, \cdots, M'$. These $\xi_{(kl)}$ are the unique solution of the following equations (Feller [1]):

$$\xi_{(kl)}=\sum_{i=1}^{M}\sum_{j=1}^{M'}\xi_{(ij)}p_{(ij)(kl)} , \tag{6}$$

$$\sum_{k=1}^{M}\sum_{l=1}^{M'}\xi_{(kl)}=1 , \qquad \xi_{(kl)}>0 , \tag{7}$$

where $k=1, 2, \cdots, M$ and $l=1, 2, \cdots, M'$.

Using (4) and (5), (6) becomes

$$\xi_{(kl)}=\lambda\sum_{i=1}^{M}\sum_{j=1}^{M'}\xi_{(ij)}\alpha_{ik}\beta_{jl}+\xi_{(kl)}-\lambda\xi_{(kl)}A_kB_l .$$

As $\lambda\neq 0$, we have

$$\xi_{(kl)}A_kB_l=\sum_{i=1}^{M}\sum_{j=1}^{M'}\xi_{(ij)}\alpha_{ik}\beta_{jl} .$$

Therefore, putting

$$\xi_{(kl)}A_kB_l=\pi_{(kl)},\ k=1, 2, \cdots, M;\ l=1, 2, \cdots, M', \tag{8}$$

we have the following equation equivalent to (6):

$$\pi_{(kl)}=\sum_{i=1}^{M}\sum_{j=1}^{M'}\pi_{(ij)}\alpha^*_{ik}\beta^*_{jl} , \tag{9}$$

where

$$\alpha^*_{ik}=\frac{\alpha_{ik}}{A_i} , \qquad \beta^*_{jl}=\frac{\beta_{jl}}{B_i} , \tag{10}$$

for $k=1, 2, \cdots, M$ and $l=1, 2, \cdots, M'$.

It is obvious from (2) and (3) that the matrices (α^*_{ik}) and (β^*_{jl}) are stochastic matrices, and each of the chains, which have the transition probabilities given by these matrices, is ergodic. Let the limiting distributions of these chains be $\{\eta^*_k\}$ and $\{\zeta^*_l\}$ respectively. These are the solutions of the equations

$$\eta_k^* = \sum_{i=1}^{M} \eta_i^* \alpha_{ik}^*, \qquad \eta_k^* > 0 \tag{11}$$

and

$$\zeta_l^* = \sum_{j=1}^{M'} \zeta_j^* \beta_{jl}^*, \qquad \zeta_l^* > 0 \tag{12}$$

respectively.

It follows that $\pi_{(kl)} = \eta_k^* \zeta_l^*$, $k=1, 2, \cdots, M$; $l=1, 2, \cdots, M'$, give a solution of the equation (9), and all $\pi_{(kl)}$ are positive. Therefore the $\eta_k^* \zeta_l^* / A_k B_l$ give a solution of (6), and all of them are positive. Consequently, if we put

$$\xi_{(kl)} = \eta_k \zeta_l, \; k=1, 2, \cdots, M; \; l=1, 2, \cdots, M', \tag{13}$$

where

$$\begin{cases} \eta_k = \dfrac{\eta_k^*/A_k}{\sum\limits_{i=1}^{M} \dfrac{\eta_i^*}{A_i}}, & (14) \\ \zeta_l = \dfrac{\zeta_l^*/B_l}{\sum\limits_{j=1}^{M'} \dfrac{\zeta_j^*}{B_i}}, & (15) \end{cases}$$

for $k=1, 2, \cdots, M$; $l=1, 2, \cdots, M'$, then the $\xi_{(kl)}$ give a solution of (6) and satisfy (7). This is the limiting distribution which is required.

To express $\xi_{(kl)}$ more explicitly we have to solve the equation (11) or (12). First we derive the $\{\eta_k^*\}$ solving the equation (11). We distinguish two cases.

Case (1°): $\qquad 0 < R \leqq \dfrac{M-1}{2}$.

In this case, we have for $1 \leqq i \leqq R$

$$\begin{cases} \alpha_{i,i+1} = \dfrac{R}{M}, \\ \alpha_{ii} = \dfrac{1}{M}, \\ \alpha_{i,i-1} = \dfrac{i-1}{M}. \end{cases}$$

Therefore we have

$$A_i = \frac{R+i}{M} \tag{16}$$

and

$$\begin{cases} \alpha^*_{i,i+1}=\dfrac{R}{R+i}\,, \\ \alpha^*_{ii}=\dfrac{1}{R+i}\,, \\ \alpha^*_{i,i-1}=\dfrac{i-1}{R+i}\,, \end{cases}$$

for $1 \leqq i \leqq R$.

For $R+1 \leqq i \leqq M-R$ we have

$$\begin{cases} \alpha_{i,i+1}=\alpha_{i,i-1}=\dfrac{R}{M}\,, \\ \alpha_{i,i}=\dfrac{1}{M}\,, \end{cases}$$

so we have

$$A_i=\frac{2R+1}{M}\,, \tag{16)'$$

and

$$\begin{cases} \alpha^*_{i,i+1}=\alpha^*_{i,i-1}=\dfrac{R}{2R+1} \\ \alpha^*_{i,i}=\dfrac{1}{2R+1}\,, \end{cases}$$

for $R+1 \leqq i \leqq M-R$.

Moreover, for $M-R+1 \leqq i \leqq M$ we have

$$\begin{cases} \alpha_{i,i+1}=\dfrac{M-i}{M}\,, \\ \alpha_{ii}=\dfrac{1}{M}\,, \\ \alpha_{i,i-1}=\dfrac{R}{M}\,. \end{cases}$$

Therefore,

$$A_i=\frac{M-i}{M+R+1-i} \tag{16)''$$

and

$$\begin{cases} \alpha^*_{i,i+1} = \dfrac{M-i}{M+R+1-i}, \\ \alpha^*_{ii} = \dfrac{1}{M+R+1-i}, \\ \alpha^*_{i,i-1} = \dfrac{1}{M+R+1-i}, \end{cases}$$

for $M-R+1 \leqq i \leqq M$.

All the α^*_{ik} with $k \neq i-1, i, i+1$ are zero.

Now the equation (11) becomes as follows:

$$\begin{cases} \eta^*_k = \dfrac{R}{R+k-1}\eta_{k-1} + \dfrac{1}{R+k}\eta^*_k + \dfrac{k}{R+k+1}\eta^*_{k+1}, \\ \text{for } 1 \leqq k \leqq R,\ \eta^*_0 \text{ being defined to be zero,} \\ \eta^*_k = \dfrac{R}{2R+1}\eta^*_{k-1} + \dfrac{1}{2R+1}\eta^*_k + \dfrac{R}{2R+1}\eta^*_{k+1}, \\ \text{for } R+2 \leqq k \leqq M-R-1, \\ \eta^*_{R+1} = \dfrac{R}{2R}\eta^*_R + \dfrac{1}{2R+1}\eta^*_{R+1} + \dfrac{R}{2R+1}\eta^*_{R+2}, \\ \eta^*_{M-R} = \dfrac{R}{2R+1}\eta^*_{M-R+1} + \dfrac{1}{2R+1}\eta^*_{M-R} + \dfrac{R}{2R}\eta^*_{M-R+1}, \\ \eta^*_k = \dfrac{M+1-k}{M+R+2-k}\eta^*_{k-1} + \dfrac{1}{M+R+1-k}\eta^*_k + \dfrac{R}{M+R-k}\eta^*_{k+1}, \\ \text{for } M-R+1 \leqq k \leqq M,\ \eta^*_{M+1} \text{ being defined to be zero.} \end{cases}$$

Solving these equations we get $\{\eta^*_k\}$ as follows. η^*_1 is undetermined.

$$\begin{cases} \eta^*_k = \eta^*_{M-k+1} = \dfrac{R+k}{R+1}\dfrac{R^{k-1}}{(k-1)!}\eta^*_1, \text{ for } 1 \leqq k \leqq R, & (17) \\ \eta^*_k = \dfrac{2R+1}{R+1}\dfrac{R^{R-1}}{(R-1)!}\eta^*_1, \text{ for } R+1 \leqq k \leqq M-R & (18) \end{cases}$$

Case (2°): $\dfrac{M-1}{2} < R \leqq M-1$.

In the same way as in the case (1°) we can rewrite the equation (11) as follows:

$$\begin{cases} \eta_k^* = \frac{R}{R+k-1}\eta_{k-1}^* + \frac{1}{R+k}\eta_k^* + \frac{k}{R+k+1}\eta_{k+1}^* , \\ \text{for } 1 \leqq k \leqq M-R-1,\ \eta_0^* \text{ being defined to be zero,} \\ \eta_{M-R}^* = \frac{R}{M-1}\eta_{M-R+1}^* + \frac{1}{M}\eta_{M-R}^* + \frac{M-R}{M}\eta_{M-R+1}^* , \\ \eta_k^* = \frac{M+1-k}{M}\eta_{k-1}^* + \frac{1}{M}\eta_k^* + \frac{k}{M}\eta_{k+1}^* , \\ \text{for } M-R+1 \leqq k \leqq R, \\ \eta_{R+1}^* = \frac{M-R}{M}\eta_R^* + \frac{1}{M}\eta_{R+1}^* + \frac{R}{M-1}\eta_{R+2}^* , \\ \eta_k^* = \frac{M+1-k}{M+R+2-k}\eta_{k-1}^* + \frac{1}{M+R+1-k}\eta_k^* + \frac{R}{M+R-k}\eta_{k+1}^* , \\ \text{for } R+2 \leqq k \leqq M,\ \eta_{M+1}^* \text{ being defined to be zero.} \end{cases}$$

Solving these equations we get $\{\eta_k^*\}$ as follows. η_1^* is undetermined.

$$\begin{cases} \eta_k^* = \eta_{M-k+1}^* = \frac{R+k}{R+1}\frac{R^{k-1}}{(k-1)!}\eta_1^*, \text{ for } 1 \leqq k \leqq M-R, \\ \eta_k^* = \frac{M}{R+1}\frac{\binom{R-1}{M-k}}{\binom{k-1}{M-R}}\frac{R^{M-R}}{(M-R)!}\eta_1^*, \text{ for } M-R+1 \leqq k \leqq R . \end{cases}$$

Now $\{\eta_k\}$ are determined as follows.

Case (1°):

From (14), (16), (16)′, (16)″, (17) and (18) we obtain

$$\eta_k = \eta_{M-k+1} = \frac{R^{k-1}}{(k-1)!}\Big/ 2\sum_{i=1}^{R}\frac{R^{i-1}}{(i-1)!} + (M-2R)\frac{R^{R-1}}{(R-1)!} \tag{19}$$

for $1 \leqq k \leqq R$,

$$\eta_k = \frac{R^{R-1}}{(R-1)!}\Big/ 2\sum_{i=1}^{R}\frac{R^{i-1}}{(i-1)!} + (M-2R)\frac{R^{R-1}}{(R-1)!} \tag{20}$$

for $R+1 \leqq k \leqq M-R$.

Just in the same way we have for

Case (2°):

$$\eta_k = \eta_{M-k+1} = \frac{R^{k-1}}{(k-1)!}\Big/ 2\sum_{i=1}^{M-R}\frac{R^{i-1}}{(i-1)!} + \left\{\sum_{i=M-R+1}^{R}\frac{\binom{R-1}{M-i}}{\binom{i-1}{M-R}}\right\}\frac{R^{M-R}}{(M-R)!} , \tag{21}$$

for $1 \leqq k \leqq M-R$,

$$\eta_k=\frac{\binom{R-1}{M-k}}{\binom{k-1}{M-R}}\frac{R^{M-R}}{(M-R)!}\Big/2\sum_{i=1}^{M-R}\frac{R^{i-1}}{(i-1)!}+\left\{\sum_{i=M-R+1}^{R}\frac{\binom{R-1}{M-i}}{\binom{i-1}{M-R}}\right\}\frac{R^{M-R}}{(M-R)!}\,, \tag{22}$$

for $M-R+1\leqq k\leqq R$.

Replacing M and R by M' and R' respectively in the above formulae we can get $\{\zeta_l\}$ as follows;

Case (1°): $\qquad 0<R'\leqq\frac{M'-1}{2}$

$$\zeta_l=\zeta_{M'-l+1}=\frac{R'^{l-1}}{(l-1)!}\Big/2\sum_{i=1}^{R'}\frac{R'^{j-1}}{(j-1)!}+(M'-2R')\frac{R'^{R'-1}}{(R'-1)!}\,, \tag{23}$$

for $1\leqq l\leqq R'$,

$$\zeta_l=\frac{R'^{R'-1}}{(R'-1)!}\Big/2\sum_{j=1}^{R'}\frac{R'^{j-1}}{(j-1)!}+(M'-2R')\frac{R'^{R'-1}}{(R'-1)!}\,, \tag{24}$$

for $R'+1\leqq l\leqq M'-R'$.

Case (2°): $\qquad \frac{M'-1}{2}<R'\leqq M'-1$

$$\zeta_l=\zeta_{M'-l+1}=\frac{R'^{l-1}}{(l-1)!}\Big/2\sum_{j=1}^{M'-R'}\frac{R'^{j-1}}{(j-1)!}+\left\{\sum_{j=M'-R+1}^{R}\frac{\binom{R'-1}{M'-j}}{\binom{j-1}{M'-R'}}\right\}\frac{R'^{M'-R'}}{(M'-R')!} \tag{25}$$

for $1\leqq l\leqq M'-R'$,

$$\zeta_l=\frac{\binom{R'-1}{M'-l}}{\binom{l-1}{M'-R'}}\frac{R'^{M'-R'}}{(M'-R')!}\Big/2\sum_{j=1}^{M'-R'}\frac{R'^{j-1}}{(j-1)!}+\left\{\sum_{j=M'-R'+1}^{R'}\frac{\binom{R'-1}{M'-j}}{\binom{j-1}{M'-R'}}\right\}\frac{R'^{M'-R'}}{(M'-R')!} \tag{26}$$

for $M'-R'+1\leqq l\leqq R'$.

In a special case $\{\eta_k\}$ or $\{\zeta_l\}$ is a binomial distribution. For instance, when $R=M-1$, the $\{\eta_k\}$ is the following binomial distribution:

$$\eta_k=\binom{M-1}{k-1}2^{-(M-1)}\,,\qquad 1\leqq k\leqq M\,.$$

From the $\{\eta_k\}$ and $\{\zeta_l\}$ so obtained, together with (13), the $\xi_{(kl)}$ can be written explicitly. This distribution is bellshaped except that in the case (1°) its shape is flat on some range, around the center, which depends on the value of R or R'. Anyway the central part of A is more probable as the position of any particle in the long run. We remark that this result is derived from the assumption of the equiprobable, that is, impartial throwing of cakes on the area A. In practical situations the throwing for the central part will be even more frequent than assumed in our model. This means that actually the central part of A is even more probable, as the position of any member of the crowd, than assumed in our model. Thus our result suggests that, in "motimaki", congestion of considerable degree will arise in central part of the area in the long run, and that this congestion is the natural consequence originating from the mechanism of the movement of the crowd.

4. Occurrence of critical level

In this section we treat the occurrence of the situation above the critical level of congestion in "motimaki". To deal with this problem we introduce another model than the one considered in the previous sections, because it is difficult to take up the problem on the basis of the previous model.

In this model we consider a partition of the area A into M vertical strips of the same size as shown in Fig. 3, and we suppose that the crowd is composed of a number of groups each of which moves about from one strip into another. We represent these strips by co-ordinate i, $i=1, 2, \cdots, M$. Moreover, we suppose that, during the movement, the original order of these groups is preserved, while it may happen that the same strip is occupied by several groups at the same time.

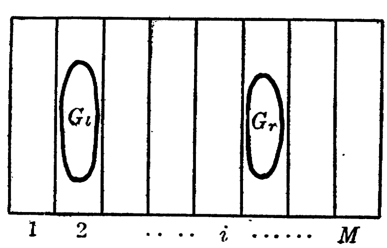

Fig. 3.

To define the critical level of congestion we consider the range of the area occupied by the total groups. This range is defined by the distance between the right-most group G_r and the left-most group G_l. We consider that the congestion of the crowd becomes heavy if the range of the groups exceeds a certain value C, and define the critical value of congestion as this value.

We define that the congestion of the crowd is above critical level if the distance between G_r and G_l exceeds C, and take up the problem of determining the mean and variance of the time of duration above or below the critical level.

Now we have to introduce some assumptions about the movement of the groups. According to the preceding formulation it is the movements of G_r and G_l that directly determine the occurrence of the situation above the critical level as considered above. We, therefore, make assumptions concerning the movements of G_r and G_l. Of course, in practice their movements will be restricted by the movements of the other groups. But we consider here rather rough and discard the exact consideration of the influence of other groups. We assume the following:

(i) The cakes are thrown on one of the strips at regular time intervals with equal probability. The consecutive throwings are independent.

(ii) M is an even integer: $M=2N$. At the beginning of the process of "motimaki", the difference between the co-ordinates of the strips, on which G_r and G_l lie, are even.

(iii) Every time the cakes are thrown the movements of G_r and G_l may occur, though it is also probable that in some throwing one or both of G_r and G_l do not move. The way of this movement is determined by the following rules (iv), (v) and (vi).

(iv) When G_r lies on a strip with the co-ordinate i, $1\leqq i\leqq M-2$, and the cakes are thrown on a strip with the co-ordinate j, $i+2\leqq j\leqq M$, G_r moves onto the strip $i+2$, while G_l does not move. Similarly, when G_l lies on a strip with the co-ordinate k, $3\leqq k\leqq M$, and the cakes are thrown on a strip with the co-ordinate l, $1\leqq l\leqq k-2$, G_l moves onto the strip $k-2$, while G_r does not move.

We add here, for simplicity, the proviso that, when G_r lies on the strip M and the cakes are thrown on the strip 1, G_l does not move, and when G_l lies on the strip 1 and the cakes are thrown on the strip M, G_r does not move.

(v) When G_r lies on a strip with the co-ordinate i, $2\leqq i\leqq M$, and the cakes are thrown on a strip which lies on the left of G_r but is not on the left of G_l, then G_r moves onto the strip $i-1$. Similarly, when G_l lies on a strip with the co-ordinate k, $1\leqq k\leqq M-1$, and the cakes are thrown on a strip which is on the right of G_l but not on the right of G_r, then G_l moves onto the strip $k+1$.

(vi) In other cases the movement of G_r or G_l does not occur in the throwing concerned.

The assumption (iv) states that, when the cakes are thrown outside the strips of G_r and G_l, only the nearer groups can move. This assumption is adopted to express roughly the disturbance of other groups lying between G_r and G_l, as well as the situation that the members of the crowd distant from the spot of falling will not participate in the movement caused by this falling. Moreover the disturbance of other groups is contained in the assumptions (iv) and (v) which state that the movement of G_r or G_l is a one-step transfer when the cakes are thrown between them, while the mevement is a two-step transfer when the cakes are thrown outside of them.

Next we determine the mean and variance of the time during which the congestion is above or below the critical level, under these assumptions. For this purpose we consider the consecutive distances between G_r and G_l in the course of the process of "motimaki". The possible values of these distances are $0, 2, 4, \cdots, 2N-2$. If we represent these by $0, 1, 2, \cdots, N-1$, then the above assumptions give a homogeneous Markov chain with the states $0, 1, 2, \cdots, N-1$ and the following transition probabilities:

$$\begin{cases} p_{k,k+1}=\dfrac{2N-2k-3}{2N}, & 0\leqq k\leqq N-2 \\ p_{k,k}=\dfrac{4}{2N}, & 1\leqq k\leqq N-2 \\ p_{k,k-1}=\dfrac{2k-1}{2N}, & 1\leqq k\leqq N-1 \\ p_{N-1,N-1}=\dfrac{3}{2N}, & \\ p_{0,0}=\dfrac{3}{2N}, & \\ p_{ij}=0 & \text{for other } i, j. \end{cases}$$

It is obvious that this chain is irreducible. Further, being a finite chain, the chain is ergodic.

First let $2k(C+1\leqq k\leqq N-1)$ be the initial distance between G_r and G_l, and we take up the time which is required for the system to attain for the first time to the state C. We denote by $u_{k,n}$ the probability that this time is equal to $n, n=1, 2, \cdots$. Then $\sum_{n=1}^{\infty} u_{k,n}=1$, because the chain is ergodic.

Now it is easily seen that the $u_{k,n}$ satisfies the following difference equation:

$$u_{k,n+1}=\frac{2N-2k-3}{2N}u_{k+1,n}+\frac{4}{2N}u_{kn}+\frac{2k-1}{2N}u_{k-1,n}\,. \tag{27}$$

This equation holds only for $C+2\leqq k\leqq N-2, n\geqq 1$. But if we put

$$u_{C,n}=0\ , \qquad n\geqq 1 \tag{28}$$

$$u_{N,n}=u_{N-1,n}\ , \qquad n\geqq 1 \tag{29}$$

$$u_{k,0}=0\ , \qquad C+1\leqq k\leqq N \tag{30}$$

$$u_{C,0}=1\ , \tag{31}$$

then (27) holds for $C+1\leqq k\leqq N-1, n\geqq 0$.

All the moments of the probability distribution $\{u_{kn}\}$ are finite, that is,

$$a_k^{(\nu)}=\sum_{n=1}^{\infty} n^\nu u_{kn}<\infty \tag{32}$$

for all integers $\nu\geqq 1$.

To see this, consider the factorial moment generating function

$$\varphi_k(s)=\sum_{n=0}^{\infty} u_{kn}s^n\ , \qquad C+1\leqq k\leqq N-1\,.$$

Then

$$\frac{d^{(\nu)}}{ds^\nu}\varphi_k(s)=\sum_{n=1}^{\infty} n(n+1)\cdots(n-\nu+1)u_{k,n}s^{n-\nu}$$

for $|s|<1, \nu\geqq 1$. In the power series of the right-hand side of this identity, all coefficients are positive. Therefore, if $\sum_{n=1}^{\infty} n(n+1)\cdots(n-\nu+1)u_{kn}=\infty$, then $\lim_{s\to 1-0}(d^{(\nu)}/ds^\nu)\varphi_k(s)=\infty$. From this it is seen that in order to prove (32) it suffices to show that $\lim_{s\to 1-0}(d^{(\nu)}/ds^\nu)\varphi_k(s)$ converges ($\nu\geqq 1$).

From (27) it is easily seen that the $\varphi_k(s)$ satisfies the following difference equation:

$$\varphi_k(s)=\left\{\frac{2N-2k-3}{2N}\varphi_{k+1}(s)+\frac{4}{2N}\varphi_k(s)+\frac{2k-1}{2N}\varphi_{k-1}(s)\right\}s \tag{33}$$

for $C+1\leqq k\leqq N-1$. From (33), (28), (29) and (31) we can show that all of the $\varphi_k(s)$ are rational functions. On the other hand $\lim_{s\to 1-0}\varphi_k(s)=\sum_{n=0}^{\infty} u_{k,n}=1$, so $s=1$ is not a pole of the rational function $\varphi_k(s)$. Consequently $\lim_{s\to 1-0}(d^{(\nu)}/ds^{\nu})\varphi_k(s)$ converges. Thus we have (32).

Now we determine the mean $m_{k,C}$ and the variance $v_{k,C}$ of the probability distribution $\{u_{kn}\}$. These are defined as

$$m_{k,C}=a_k^{(1)}=\sum_{n=1}^{\infty} nu_{kn}\ ,$$

$$v_{k,C}=a_k^{(2)}-\{a_k^{(1)}\}^2=\sum_{n=1}^{\infty} n^2u_{kn}-m_{k,C}^2\ .$$

For the $m_{k,C}$ we have, from (27) together with the relation $\sum_{n=1}^{\infty} u_{kn}=1$, the following difference equation:

$$m_{k,C}=\frac{2N-2k-3}{2N}(m_{k+1,C}+1)+\frac{4}{2N}(m_{k,C}+1)+\frac{2k-1}{2N}(m_{k-1,C}+1) \tag{34}$$

for $C+1\leqq k\leqq N-1$.

Moreover, from (29) we have

$$m_{N,C}=m_{N-1,C}\ . \tag{35}$$

From (34) we have

$$(2N-2k-3)(m_{k+1,C}-m_{k,C})+2N=(2k-1)(m_{k,C}-m_{k-1,C}) \tag{36}$$

for $C+1\leqq k\leqq N-1$.

Putting

$$m_{k+1,C}-m_{k,C}=\alpha_k\ , \qquad C\leqq k\leqq N-1\ , \tag{37}$$

we obtain from (36) and (35)

$$(2N-2k-3)\alpha_k+2N=(2k-1)\alpha_{k-1} \tag{38}$$

for $C+1\leqq k\leqq N-1$,

$$\alpha_{N-1}=0\ . \tag{39}$$

From (38) and (39) we have

$$\alpha_{N-2}=\frac{2N}{2N-3}\ . \tag{40}$$

Now define β_k, $C\leqq k\leqq N-2$ by the following relation:

$$\alpha_k=\frac{(2N-2k-5)(2N-2k-7)\cdots 531}{(2N-3)(2N-5)\cdots(2k+3)(2k+1)}\beta_k\ . \tag{41}$$

Here, in the case of $k=N-2$, the numerator of the right-hand of (41) is to be interpreted as unity.

From (40) and (41) we have

$$\beta_{N-2}=2N. \tag{42}$$

Substitute (41) for α_k in (38). Then (38) becomes, after a simple calculation, the following relation:

$$\beta_k+2N\frac{\binom{2N-2}{2k}}{\binom{N-1}{k}}=\beta_{k-1}, \tag{43}$$

$C+1\leqq k\leqq N-2.$

Adding the both sides of this equality with respect to k ranging from $C+1$ to $N-2$, we obtain

$$\beta_{N-2}+2N\sum_{k=C+1}^{N-2}\frac{\binom{2N-2}{2k}}{\binom{N-1}{k}}=\beta_C.$$

So by (42) we have

$$\beta_C=2N\sum_{k=C+1}^{N-1}\frac{\binom{2N-2}{2k}}{\binom{N-1}{k}}. \tag{44}$$

From (43) and (44) we obtain

$$\beta_k=2N\sum_{j=k+1}^{N-1}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}, \tag{45}$$

for $C\leqq k\leqq N-2.$

Substituting (45) for β_k in (41), we have, after a simple calculation,

$$\alpha_k=\frac{2N}{2N-3}\frac{\binom{N-2}{k}}{\binom{2N-4}{2k}}\sum_{j=k+1}^{N-1}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}, \tag{46}$$

$C\leqq k\leqq N-2.$

From (37) and (46), and noting $m_{C,C}=0$ because of (28), we finally obtain

$$m_{k,C}=\sum_{i=C}^{k-1}\frac{2N}{2N-3}\frac{\binom{N-2}{i}}{\binom{2N-4}{2i}}\sum_{j=i+1}^{N-1}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,, \tag{47}$$

$C+1\leqq k\leqq N-1$.

As the special case of $k=C+1$, we have

$$m_{C+1,C}=\frac{2N}{2N-3}\frac{\binom{N-2}{C}}{\binom{2N-4}{2C}}\sum_{j=C+1}^{N-1}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,. \tag{48}$$

This is the mean of the time during which the congestion is below the critical level.

We can similarly deal with $a_k^{(2)}$. For these we have, from (27) together with the relation $\sum_{n=1}^{\infty}u_{kn}=1$, the following difference equation:

$$a_k^{(2)}=\frac{2N-2k-3}{2N}(a_{k+1}^{(2)}+2m_{k+1,C}+1)+\frac{4}{2N}(a_k^{(2)}+2m_{k,C}+1)$$
$$+\frac{2k-1}{2N}(a_{k-1}^{(2)}+2m_{k-1,C}+1)\,, \tag{49}$$

for $C+1\leqq k\leqq N-1$.

Moreover, from (29) we have

$$a_N^{(2)}=a_{N-1}^{(2)}\,. \tag{50}$$

From (49) we obtain

$$(2N-2k-3)\alpha_k+2\{(2N-2k-3)m_{k+1,C}+4m_{k,C}+(2k-1)m_{k-1,C}\}$$
$$+2N=(2k-1)\alpha_{k-1}\,, \tag{51}$$

for $C+1\leqq k\leqq N-1$, where this time α_k are defined by

$$\alpha_k=a_{k+1}^{(2)}-a_k^{(2)} \tag{52}$$

for $C\leqq k\leqq N-1$.

Now define β_k by (41), and put these into (51). Then we obtain

$$\beta_k+B_k\frac{\binom{2N-2}{2k}}{\binom{N-1}{k}}=\beta_{k-1} \tag{53}$$

for $C+1\leqq k\leqq N-2$, where the definition of B_k is as follows:

$$B_k=2\{(2N-2k-3)m_{k+1,C}+4m_{k,C}+(2k-1)m_{k-1,C}+N\}\,, \tag{54}$$

for $C+1 \leqq k \leqq N-1$.

By using (34) we can give B_k the following simple form:

$$B_k = 2N(2m_{k,C}-1)\,, \qquad \text{for } C+1 \leqq k \leqq N-1\,. \tag{55}$$

From (53) we have

$$\beta_{N-2} + \sum_{k=C+1}^{N-2} B_k \frac{\binom{2N-2}{2k}}{\binom{N-1}{k}} = \beta_C\,. \tag{56}$$

On the other hand, from (50) and (52) we have $\alpha_{N-1}=0$. Therefore, from (51) and the definition of β_{N-2} we obtain

$$\beta_{N-2} = (2N-3)\alpha_{N-2} = 2\{-m_{N,C} + 4m_{N-1,C} + (2N-3)m_{N-2,C} + N\}\,.$$

From this and the definition (54) of B_{N-1}, we obtain

$$\beta_{N-2} = B_{N-1}\,. \tag{57}$$

From (56) and (57) we have

$$\beta_C = \sum_{k=C+1}^{N-1} B_k \frac{\binom{2N-2}{2k}}{\binom{N-1}{k}}\,. \tag{58}$$

On the other hand, from (53) we have

$$\beta_k + \sum_{j=C+1}^{k} B_j \frac{\binom{2N-2}{2j}}{\binom{N-1}{j}} = \beta_C\,, \tag{59}$$

for $C+1 \leqq k \leqq N-2$.

Putting (58) into (59) we have

$$\beta_k = \sum_{j=k+1}^{N-1} B_j \frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,, \tag{60}$$

for $C \leqq k \leqq N-2$.

Substituting (60) for β_k in (41) and, after a simple calculation, we obtain

$$\alpha_k = \frac{1}{2N-3} \frac{\binom{N-2}{k}}{\binom{2N-4}{2k}} \sum_{j=k+1}^{N-1} B_j \frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,, \tag{61}$$

for $C \leqq k \leqq N-2$.

From (52) and (61), and noting $a_C^{(2)}=0$ because of (28), we finally obtain

$$a_k^{(2)}=\frac{1}{2N-3}\sum_{i=C}^{k-1}\frac{\binom{N-2}{i}}{\binom{2N-4}{2i}}\sum_{j=j+1}^{N-1}B_j\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,, \tag{62}$$

for $C+1\leqq k\leqq N-1$, where B_j's are given by (55).

Using (47), we can express $a_k^{(2)}$ as follows:

$$a_k^{(2)}=\left\{\frac{4N}{2N-3}\sum_{i=C}^{k-1}\frac{\binom{N-2}{i}}{\binom{2N-4}{2i}}\sum_{j=i+1}^{N-1}m_{j,C}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\right\}-m_{k,C}\,. \tag{63}$$

The variance of the distribution $\{u_{kn}\}$ is given by

$$v_{k,C}=\left\{\frac{4N}{2N-3}\sum_{i=C}^{k-1}\frac{\binom{N-2}{j}}{\binom{2N-4}{2i}}\sum_{j=i+1}^{N-1}m_{j,C}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\right\}-m_{k,C}(1+m_{k,C})\,, \tag{64}$$

for $C+1\leqq k\leqq N-1$.

As the special case of $k=C+1$, we have

$$v_{C+1,C}=\left\{\frac{4N}{2N-3}\frac{\binom{N-2}{C}}{\binom{2N-4}{2C}}\sum_{j=C+1}^{N-1}m_{j,C}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\right\}-m_{C+1,C}(1+m_{C+1,C})\,. \tag{65}$$

This is the variance of the time during which the congestion is below the critical level.

Substituting C in (48) and (65) with $N-C-2$ respectively we obtain the mean m_C^* and the variance v_C^* of the time during which the congestion is above the critical level, because of the symmetry of the transition probabilities concerned. Thus we have

$$m_C^*=\frac{2N}{2N-3}\frac{\binom{N-2}{C}}{\binom{2N-4}{2C}}\sum_{j=N-C-1}^{N-1}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\,, \tag{66}$$

$$v_C^*=\left\{\frac{4N}{2N-3}\frac{\binom{N-2}{C}}{\binom{2N-4}{2C}}\sum_{j=N-C-1}^{N-1}m_{j,N-C-2}\frac{\binom{2N-2}{2j}}{\binom{N-1}{j}}\right\}-m_C^*(1+m_C^*)\,, \tag{67}$$

where m_{ij}'s are given by (47).

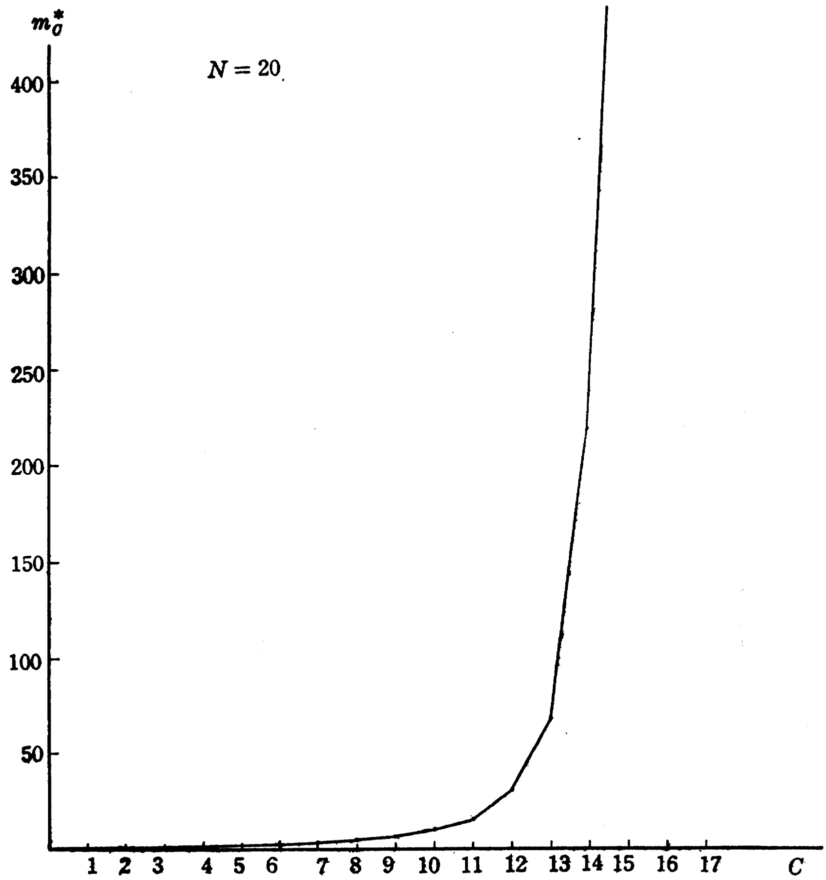

Fig. 4.

As an example we show in Fig. 4 the relation between C and m_C^* for $N=20$.

In practical situations pertaining to the process of "motimaki", it is important to foresee the danger, that may be caused by the congestion of the crowd concerned, from the degree of compactness of this crowd in the area of the process. From a certain standpoint the result of this section is applicable for this purpose. We give next the rough description of this application.

We represent the degree of the possible danger of the process by the expected duration m_C^* of the critical level. Here the critical value C is considered to have some relation with the density of the crowd in the area as a whole. Let this relation be such that C is proportionate to this density of the crowd. Then the relation between C and m_C^* expresses the relation between the density of the crowd and the degree of the possible danger.

Thus, using this relation, we shall be able to make some forecasting about the possible danger of the process. For example, in the case shown in Fig. 4, we forecast that the danger of the process increases rapidly if the density of the crowd exceed the value which corresponds to $C=13$.

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Reference

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