DIFFUSION PROCESS CORRESPONDING TO $\frac{1}{2}\sum\frac{\partial^2}{\partial x^{i2}}+\sum b^i(x)\frac{\partial}{\partial x^i}$

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1. Introduction and summary

In this paper, we shall construct the diffusion process corresponding to the operator $A=1/2\sum\partial^2/(\partial x^i)^2+\sum b^i(x)\partial/(\partial x^i)$ in the N-dimensional space, where $b^i(X)$'s are bounded uniformly continuous functions on the whole space, and investigate properties of the semi-group associated with this process.

In section 3, we construct the process in the space of continuous paths by using the stochastic integral. It is shown that the process constructed is absolutely continuous with respect to the Brownian motion and vice versa, so far as the paths during a finite time are concerned. The method of construction is essentially the same as in [1] although this paper treats the one dimentional case.

In section 4, it is shown that the semi-group associated with this process can be restricted to certain functional spaces (for example, the space of continuous functions vanishing at infinity) and is strongly continuous on these spaces. These spaces are normed by the maximum absolute value.

In section 5, assuming the Lipschitz condition for $b^i(x)$'s, the Hille-Yoshida's generator of the semi-group with its domain is investigated, which is a closed extension of A in a certain sense. Especially Dynkin's generator of this process is represented in the form independent of the process.

In section 6, some probabilistic properties of this process are stated which are immediately deduced from the fact that this process and the Brownian motion process are absolutely continuous with respect to each other.

The applications (for example, to Dirichlet's problem to the operator A) will be treated elsewhere.

2. Notations

In this section we summarize the notations which are used in the present paper.

R^N : the N-dimensional Euclidian space.

$\mathfrak{B}^N$: the Borel field generated by the open sets in R^N.

B : the set of real valued and bounded $\mathfrak{B}^N$-measurable functions on R.

C : the set of real valued and bounded continuous functions on R^N.

$\bar{C}$: the set of uniformly continuous functions in C.

C_∞ : the set of functions in C, each of which has a (finite) limit at infinity.

C_0 : the set of functions in C_∞ whose limits at infinity are zero.

W : the space of continuous functions from $[0, \infty)$ into R^N.

w : the element of W

$x(t, w)$: the value of w at $t : t \in [0, \infty)$.

w_t^+ : the shifted path of $w : x(s, w_t^+) = x(t+s, w)$

w_t^- : the stopped path of $w : x(s, w_t^-) = x(\min(s, t), w)$

$\mathfrak{B}_\infty$: the Borel field generated by the subsets in W such as $\{w : x(t, w) \in A \text{ for any } t \in [0, \infty)\}$ for $A \in \mathfrak{B}^N$.

$\mathfrak{B}_t$: the Borel field generated by the subsets in W such as $\{w : x(s, w) \in A\}$ for $s \leqq t$, $A \in \mathfrak{B}^N$.
or $\mathfrak{B}_t = \{B = \{w : w_t^- \in B'\} \text{ for all } B' \in \mathfrak{B}_\infty\}$.

σ (Markov time) : the positive $\mathfrak{B}_\infty$-measurable function on W such as $\{w : \sigma(w) < t\} \in \mathfrak{B}_t$, w_σ^+ and w_σ^- are defined as w_t^+ and w_t^-.
$\mathfrak{B}_{\sigma+} = \bigcap_{\varepsilon \downarrow 0} \mathfrak{B}_{\sigma+\varepsilon} = \bigcap_{\varepsilon \downarrow 0} \{B : B = \{w : w_{\sigma+\varepsilon}^- \in B'\} \text{ for all } B' \in \mathfrak{B}_\infty\}$. (Then, σ and $x(\sigma, w)$ are B-measurable).

$\{\bar{P}_x\}$, $x \in R^N$: the Brownian measures on $(W, \mathfrak{B}_\infty)$ i.e.,

$$P(\{w : x(t_i, w) \in A_i \; i = 1, 2, \cdots n\})$$
$$= \int_{A_1} \cdots \int_{A_n} \frac{\exp\left\{-\dfrac{\sum (y_n^i - y_{n-1}^i)^2}{2(s_n - s_{n-1})} - \cdots - \dfrac{\sum (y_1^i - x^i)^2}{2s_1}\right\}}{\sqrt{(2\pi)^{nN}(s_n - s_{n-1}) \cdots (s_2 - s_1)s_1}} dy_1 \cdots dy_n$$

where $A_i \in \mathfrak{B}^N$.

$E\{\cdot\}$: the expectation with respect to P-measure. (For $\bar{P}_x$-measure or P_x-measure, we shall write their expectations $\bar{E}_x\{\cdot\}$ or $E_x\{\cdot\}$ etc.) $E\{\cdot \mid \mathfrak{B}\}$ or $P\{\cdot \mid \mathfrak{B}\}$ are used for the conditional expectation or the conditional probability with respect to Borel field $\mathfrak{B}$,

$\chi(A)$: the characteristic function of the set A.

3. Construction of the process.

Let $\boldsymbol{b}(x)=(b^1(x), \cdots, b^N(x))$ be a vector valued function on R^N such that $b^i(x) \in \bar{C}$. Set

$$J_x(t, w)=J_x(t, w, \boldsymbol{b})=\int_0^t \sum b^i(x(t))\, dx^i(t) \tag{3.1}$$

$$K(t, w)=K(t, w, \boldsymbol{b})=\frac{1}{2}\int_0^t (\sum b^i(x(t))^2)\, dt \tag{3.2}$$

$$I_x(t, w)=I_x(t, w, \boldsymbol{b})=J_x(t, w, \boldsymbol{b})+K(t, w, \boldsymbol{b}) \tag{3.3}$$

$$F_x(t, w)=F_x(t, w, \boldsymbol{b})= \exp (I_x(t, w, \boldsymbol{b})) \tag{3.4}$$

where the integral in (3.1) is defined as a stochastic integral (c.f. [2]) (therefore it can be determined except $\bar{P}_x$-measure zero). Thus, we have the following proposition (by theorem 1.1 in [2]).

PROPOSITION 3.1. *$F(t, w)$ is a stochastic integral, and*

$$dF(t, w)=F(t, w) \sum b^i(x(t))\, dx^i(t) .$$

As a preparation for theorem 3.4, we shall prove some lemmas and propositions.

LEMMA 1. *Let $\{f_n(w, \lambda)\}$ and $g(w, \lambda)$ be random variables with the parameter λ. If $E\{f_n(w, \lambda)^2\} \leqq K$ where K is independent of n and λ, and moreover, if*

$$\lim_{n\to\infty} P\{| f_n-g | \geqq\varepsilon\} =0$$

uniformly with respect λ, then we have

$$\lim_{n\to\infty} E\{| f_n-g |\} =0 .$$

PROOF. From $E\{f_n^2\} \leqq K$, where K is a constant independent of n and λ, by choosing a suitable subsequence we have

$$E\{g^2\} \leqq \underline{\lim}\, E\{f_{n_j}^2\} \leqq K .$$

Therefore, for any $\varepsilon>0$, we can take a positive number n_0 independent of λ such that

$$P\left\{| f_n-g | >\frac{\varepsilon}{2}\right\}\leqq\frac{\varepsilon^2}{16K} \quad \text{for} \quad n\geqq n_0(\varepsilon) .$$

$$E\{|f_n-g|\}=E\Big\{\chi\Big(|f_n-g|\leqq\frac{\varepsilon}{2}\Big)|f_n-g|\Big\}^*$$
$$+E\Big\{\chi\Big(|f_n-g|>\frac{\varepsilon}{2}\Big)|f_n-g|\Big\}$$
$$\leqq\frac{\varepsilon}{2}+\sqrt{P\Big\{|f_n-g|>\frac{\varepsilon}{2}\Big\}E\{|f_n-g|^2\}}$$
$$\leqq\frac{\varepsilon}{2}+\sqrt{\frac{\varepsilon^2}{16K}\cdot 4K}\leqq\varepsilon$$

for any $n\geqq n_0$. q.e.d.

PROPOSITION 3.2. *Let* $0=t_0<t_1<\cdots<t_{n-1}<t_n=t$,

$$J_n(t,w)=\sum_{k=0}^{n-1}\Big\{\sum_{i=1}^{N}b^i(x(t_k))(x^i(t_{k+1})-x^i(t_k))\Big\},$$

and $\|\boldsymbol{b}\|=\sup_{x,i}|b^i(x)|$. Then we have

$$\bar{E}(J_n(t,w)^{2p})\leqq\frac{(2p)!}{p!}(N\|\boldsymbol{b}\|^2t)^p. \tag{3.5}$$

PROOF. We shall give the proof by induction with respect to n. When $n=1$, (3.5) is easily obtained by evaluating the $2p$-th moment of the Gaussian distribution, for $b^i(x(0))$'s are constants. Now, assume (3.5) holds for $n=m$. Then

$$\bar{E}_x(J_{m+1}(t,w)^{2p})=\sum_{r=0}^{2p}\binom{2p}{r}\bar{E}_x[J_m^{2p-r}(t_m)\{\sum b^i(x(t_m))(x^i(t_{m+1})-x^i(t_m))\}^r].$$

where

$$\bar{E}_x\Big[J_m^{2p-r}(t_m)\Big\{\sum_i b^i(x(t_m))(x^i(t_{m+1})-x^i(t_m))\Big\}^r\Big]$$
$$=\bar{E}_x[J_m^{2p-r}(t_m)\bar{E}_x\{(\sum b^i(x(t_m))(x^i(t_{m+1})-x^i(t_m)))^r\mid\mathfrak{B}_{t_m}\}]$$
$$=\begin{cases}0 & \text{if } r \text{ is odd.}\\ \bar{E}_x\Big[J_m^{2p-2s}(t_m)\dfrac{(2s)!}{s!}\{\sum b^i(x(t_m))^2\}^s(t_{m+1}-t_m)^s\Big] & \text{if } r=2s.\end{cases}$$

Therefore, we have

$$\bar{E}_x\{J_{m+1}^{2p}(t)\}\leqq\sum_{s=0}^{p}\frac{(2p)!}{(2p-2s)!\,(2s)!}\frac{(2s)!}{s!}\{(t_{m+1}-t_m)N\|\boldsymbol{b}\|^2\}^s\bar{E}_x(J_m^{2p-2s})$$
$$\leqq\sum_{s=0}^{p}\frac{(2p)!}{(2p-2s)!\,(2s)!}\frac{(2s)!}{s!}\{(t_{m+1}-t_m)N\|\boldsymbol{b}\|^2\}^s\frac{(2p-2s)!}{(p-s)!}(N\|\boldsymbol{b}\|^2t_m)^{p-s}$$

* $\chi(A)$ means a characteristic function of a set A.

$$=\frac{(2p)!}{p!}(N\,\|\,\boldsymbol{b}\,\|^2\,t_{m+1})^p \qquad \text{where } t_{m+1}=t\ .\ \text{q.e.d.}$$

PROPOSITION 3.3. *Employing the same notations as in proposition 3.2, for any $p>0$ we have*

$$\bar{E}_x\{(\exp J_n(t,\,w))^p\}\leqq 2e^{p^2N\|\boldsymbol{b}\|^2t} \qquad p=0,\ \pm 1,\ \cdots \quad (3.6)$$

$$\bar{E}_x\{(\exp J_x(t,\,w))^p\}\leqq 2e^{p^2N\|\boldsymbol{b}\|^2t} \qquad p=0,\ \pm 1,\ \cdots \quad (3.7)$$

and if $\sup_k |t_{k+1}-t_k|\to 0$, *then* $\exp J_n(t,\,w)$ *converges to* $\exp J_x(t,\,w)$ *in the* $L^p(W,\,d\bar{P}_x)$ *sense* $(p>0)$ (see [2]).

PROOF. By proposition 3.2,

$$\begin{aligned}&\bar{E}_x\{(\exp J_n(t,\,w))^p\}\\ &=\bar{E}_x\{\exp J_n(t,\,w,\,p\boldsymbol{b})\}\leqq E\{\exp J_n(t,\,w,\,p\boldsymbol{b})\}+E_x\{\exp -J_n(t,\,w,\,p\boldsymbol{b})\}\\ &\leqq 2\sum_{s=0}^{\infty}\frac{1}{2s!}\bar{E}_x(J_n(t,\,w,\,p\boldsymbol{b})^{2s})\\ &\leqq 2\sum_{s=0}^{\infty}\frac{1}{(2s)!}\left(\frac{(2s)!}{s!}(Np^2\,\|\,\boldsymbol{b}\,\|^2\,t)^s\right)=2e^{p^2N\|\boldsymbol{b}\|^2t}\end{aligned}$$

On the other hand, if $\sup_k |t_{k+1}-t_k|$ converges to zero, then $\{J_n(t,\,w)\}$ converges to $J_x(t,\,w)$ in $L^2(W,\,d\bar{P}_x)$-sense (see [2]). Therefore, $\{(\exp J_n(t,\,w))^p\}$ converges to $(\exp J_x(t,\,w))^p$ in probability.

Using the Fatou's theorem for a suitable subsequence $\{n_j\}$ we get

$$\bar{E}_x\{(\exp J_x(t,\,w))^p\}\leqq \lim \bar{E}_x\{(\exp J_{n_j}(t,\,w))^p\}\leqq 2e^{p^2N\|\boldsymbol{b}\|^2t}\ .$$

The latter part of the proposition is easily proved by lemma 1. q.e.d.

THEOREM 3.4. *Employing the same notations as in proposition* 3.2, *we have*

$$\bar{E}_x(F_x(t,\,w)^p)\leqq 2e^{3/2p^2N\|\boldsymbol{b}\|^2t}(p=0,\ \pm 1,\ \cdots) \qquad (3.8)$$

and if $\sup |t_{k+1}-t_k|\to 0$, *then* $\exp(I_n(t,\,w))$ *converges to* $F_x(t,\,w)$ *in* $L^p(W,\,d\bar{P}_x)$*-sense,* $(p=1,\,2,\,\cdots)$, *where* $I_n(t,\,w)=J_n(t,\,w)+K(t,\,w)$.

PROOF. Since $\frac{1}{2}\sum b^i(x(t))^2\leqq \frac{1}{2}N\,\|\,\boldsymbol{b}\,\|^2$, we have

$$|\,K(t,\,w)|\leqq \frac{1}{2}N\,\|\,\boldsymbol{b}\,\|^2\,t \qquad \text{for arbitrary } w.$$

Thus, by (3.6)

$$\bar{E}_x\{(\exp I(t,\,w))^p\}\leqq \bar{E}_x\{\exp J_n(t,\,w)\}\,e^{(|p|/2)N\|\boldsymbol{b}\|^2t}\leqq 2e^{3/2p^2N\|\boldsymbol{b}\|^2t} \qquad (3.9)$$

In the similar way, we get (3.8) from (3.7). As $\exp(I_n(t, w))$ converges to $F_x(t, w)$ in probability when $\sup_k |t_{k+1}-t_k|$ converges to zero, we can apply lemma 1 by (3.9) to proving the last statement in the theorem. q.e.d.

REMARK: If $I^*(t)=\int_0^t \sum b^i(s, w)\,dx^i(s)+\int_0^t c(s, w)\,ds$ where $b^i(s)$'s and $c(s)$ are $\mathfrak{B}_s$-measurable, and $\sup_{i,s}|b^i(s)|$ and $\sup_s |c(s)|<k$, then we can prove

$$\bar{E}_x(e^{|I(t)|})\leqq 2e^{3/2Nk^2t} .$$

The proof runs in the same way as in propositions 3.2, 3.3 and theorem 3.4.

We prepare the next proposition for proving the Markovian property of the process in definition 3.2.

PROPOSITION 3.5. *For any* $0<s<t<\infty$ *and* $f\in B$, *we have*

$$\bar{E}_x\{F_x(s+t, w)f(x(s+t))\mid \mathfrak{B}_s\}=F_x(s, w)\bar{E}_{x(s)}\{F_{x(s)}(t, w)f(x(t))\} \tag{3.10}$$

PROOF. Let $0=t_0<t_1<\cdots<t_m=s<t_{m+1}<\cdots<t_{m+n}=s+t$, and $t_{m+l}-s=s_l(l>0)$. Then

$$\begin{aligned}J_{n+m}(s+t)&=J_m(s)+\sum_{k=m}^{m+n-1}\sum_i b^i(x(t_k))(x^i(t_{k+1})-x^i(t_k))\\&=J_m(s)+\sum_{l=0}^{n-1}\sum_l b^i(x(s+s_l))(x^i(s+s_{l+1})-x^i(s+s_l))\\&=J_m(s)+\sum_{l=0}^{n-1}\sum b^i(x(s_l, w_s^+))(x^i(s_{l+1}, w_s^+)-x^i(s_l, w_s^+))\\&=J_m(s, w)+J_n(t, w_s^+) .\end{aligned}$$

As is easily seen

$$K(s+t, w)=K(s, w)+K(t, w_s^+)$$

and

$$K(s, w) \text{ and } J_n(s, w) \text{ are } \mathfrak{B}_t\text{-measurable,}$$

so we have

$$\begin{aligned}&\bar{E}_x\{\exp(J_{n+m}(s+t)+K(s+t))\cdot f(x(s+t))\}\\&=\exp(J_n(s, w)+K(s, w))\bar{E}_{x(s)}\{\exp(J_n(t, w)+K(t, w))f(t))\}\end{aligned} \tag{3.11}$$

by the Markovian property of the Brownian measure.

If $\sup|t_{k+1}-t_k|$ converges to zero, then a suitable subsequence of $J_m(s, w)$ converges to $J_x(s, w)$ almost surely. By this fact and theorem 3.4, we get (3.10) from (3.11). q.e.d.

Now we can define a process on $\mathfrak{B}_t$.

DEFINITION. We define a system of measures $P_{x,t}$ on $\mathfrak{B}_t$ by

$$P_{x,t}(B)=\int_B F_x(t, w)\, d\bar{P}_x \qquad \text{for } B\in\mathfrak{B}_t\,. \tag{3.12}$$

By theorem 3.4, the integral in the definition is finite.

PROPOSITION 3.6.

$$P_{x,t}\geqq 0 \tag{3.13}$$

$$P_{x,t}(W)=1\,. \tag{3.14}$$

For $s<t$ and $B\in\mathfrak{B}_s$ we get

$$P_{x,s}(B)=P_{x,t}(B)\,. \tag{3.15}$$

PROOF. 3.13 is obvious from the definition. As to (3.14), by proposition 3.1 we have

$$F_x(t, w)=\int_0^s F_x(s, w)\sum b^i(x(s))\, dx^i(s)+F_x(0, w)$$

where $F_x(0, w)\equiv 1$, and $F_x(t, w)\in L^2(W, \bar{P}_x)$. Therefore, by [1] we have $\bar{E}_x(F_x(t, w))=1$. For (3.15), taking $f\equiv 1$ in proposition 3.6, we get

$$\begin{aligned}P_{x,t}(B)&=\bar{E}_x\{\chi(w\in B)F_x(t, w)\}\\&=\bar{E}_x[\chi(w\in B)\bar{E}_x\{t, w)\mid\mathfrak{B}_t\}]\\&=\bar{E}_x\{\chi(w\in B)F_x(s, w)E_{x(s)}(t-s, w)\}\,.\end{aligned}$$

Since $\bar{E}_{x(s)}(F_{x(s)}(t-s, w))\equiv 1$ by (3.14), we have

$$\begin{aligned}P_{x,t}(B)&=\bar{E}_x\{\chi(w\in B)F_x(s, w)\}\\&=P_{x,s}(B)\,.\end{aligned}$$

q.e.d.

THEOREM 3.7. *There exists a unique system of Markovian probability measures $\{P_x\}(x\in R^N)$ on $\mathfrak{B}_\infty$ such that*

$$\textit{for}\quad B\in\mathfrak{B}_t\quad P_x(B)=\bar{E}_x(F_x(t, w)\colon B)=P_{x,t}(B) \tag{3.16}$$

P_x and $\bar{P}$ are absolutely continuous with respect to each other if they are restricted on $\mathfrak{B}_t$.

For some countable dense subset S in $[0, \infty)$, we define $\overline{W}$ as the set of all functions $\bar{x}(t, w)=\bar{w}$ from $[0, \infty)$ to R^N which satisfy

$$\varliminf_{\substack{s\to t\\ s\in S}}\bar{x}^i(s, w)\leqq\bar{x}^i(t, w)\leqq\varlimsup_{\substack{s\to t\\ s\in S}}\bar{x}^i(s, w)$$

for any t and i. Then $\overline{\mathfrak{B}}_t$ is a Borel field generated by cylinder sets

$\{w: \bar{x}(s) \in A\}$ $0 \leq s \leq t$ where $A \in \mathfrak{B}^N$, and $\overline{\mathfrak{B}}_\infty$ is a Borel field generated by sets $\{w: \bar{x}(s) \in A\}$, $0 \leq s < \infty$.

Now, by the Kolmogorov's extension theorem [3] and the Doob's existence theorem of the separable version [4], the relation (3.14) implies that

(i) there exists a countable dense subset S in $[0, \infty)$,

and

(ii) there exists a Probability measure $\bar{P}_x$ on $\overline{\mathfrak{B}}_\infty$ such that

$$\bar{P}_x(\{\bar{w}: \bar{x}(t_i) \in A_i, \overset{i=1,2,\cdots,n}{t_i \leq t}\}) = P_{xt}(\{w: x(t_i) \in A_i, \overset{i=1,2,\cdots,n}{t_i \leq t}\}) \tag{3.17}$$

for any $A_i \in \mathfrak{B}^N$.

Let $\overline{W}_t$ be a subset in $\overline{W}$ whose elements are continuous in $[0, t]$. Then $\overline{W}_t \in \overline{\mathfrak{B}}_t$ and $W = \bigcap_{t_n \uparrow \infty} \overline{W}_{t_n} \in \overline{\mathfrak{B}}_\infty$. On the other hand, by (3.14) and (3.17) we have

$$\bar{P}_x(\overline{W}_t) = P_{x,t}(W) = 1 \text{ ,}$$

Terefore

$$\bar{P}_x(W) = 1 \text{ or } \bar{P}_x(\overline{W} - W) = 0 \tag{3.18}$$

This shows the existence of the probability measure P_x on $(W, \mathfrak{B}_\infty)$ such that

$$P_x(\{w: x(t_i) \in A_i; \overset{i=1,2,\cdots n}{t_i \leq t}\}) = P_{x,t}(\{w: x(t_i) \in A_i: \overset{i=1,2,\cdots n}{t_i \leq t}\})$$

for any $A_i \in \mathfrak{B}^N$, or

$$P_x\{B\} = P_{x,t}(B) \qquad \text{for any } B \in \mathfrak{B}_t \text{ .}$$

The uniqueness of P_x is obvious.

Since $0 < F_x(t, w) < \infty$ almost surely with respect to $\bar{P}_x$ meaure, $P_{x,t}$ and $\bar{P}_x$ are absolutely continuous with respect to each other on $\mathfrak{B}_t$ by definition, that is, P_x and $\bar{P}_x$ are absolutely continous with respect to each other, if they are restricted on $\mathfrak{B}_t$.

Now, for any $s, t > 0$ and any $B \in \mathfrak{B}_s$,

$$\begin{aligned} E_x(\chi(B)f(x(s+t))) &= \bar{E}_x(\chi(B)F_x(t+s)f(x(s+t))) \\ &= \bar{E}_x(\chi(B)\bar{E}_x\{F_x(t+s)f(x(s+t))|\mathfrak{B}_s\}) \\ &= \bar{E}_x(\chi(B)F_x(s)\bar{E}_{x(s)}\{F(t)f(x(t))\}) \\ &= E_x(\chi(B)E_{x(s)}\{f(x)\}) \end{aligned}$$

by proposition 3.5.

Namely

$E_x(f(x(s+t)) \mid \mathfrak{B}_s) = E_{x(s)}(f(x(t))$ almost surely.

This is a Markovian property of the system $\{P_x\}$. q.e.d.

DEFINITION. We say that the process $(P_x, W, \mathfrak{B}_\infty)$ is a Markovian process corresponding to the generator

$$A = \frac{1}{2} \sum \frac{\partial^2}{\partial x^{i2}} + \sum b^i(x) \frac{\partial}{\partial x_i} .$$

4. Properties of the semi-group corresponding to $(P_x\ W\ \mathfrak{B}_\infty)$

For $f \in B$, we define

$$T_t f(x) = E_x f(x(t)) = \bar{E}_x(F(t) f(x(t)) . \tag{4.1}$$

Then, by theorem 3.7 we have

PROPOSITION 4.1. *T_t is a mapping from B into B such that*

(i) *if $f \geqq 0$, then $T_t f \geqq 0$*

(ii) $T_t 1 = 1$

(iii) $T_{t+s} = T_t T_s$.

In this section we shall investigate the semi-group $\{T_t\}$.

PROPOSITION 4.2. *If $f \in C$, then*

$$\lim_{t \downarrow 0} T_t f(x) = f(x) \qquad \text{for any } x \in R.$$

If $f \in \bar{C}$, then the convergence is uniform with respect to x, i.e. $\lim_{t \downarrow 0} \sup_x | T_t f(x) - f(x) | = 0$.

PROOF.

$$T_t f(x) - f(x) = E_x(f(x(t))) - \bar{E}_x(f(x(t))) + \bar{E}_x(f(x(t))) - f(x) .$$

In the first place,

$$\bar{E}_x(f(x(t))) - f(x) = \int_{R^N} dy' \cdots dy^N \frac{e^{-\sum \frac{(y^i - t^i)^2}{2t}}}{\sqrt{2\pi t^N}} (f(y) - f(x))$$

converges to zero if $f \in C$, and moreover, the convergence is uniform if $f \in \bar{C}$.

On the other hand,

$$| (E_x f(x(t))) - \bar{E}_x(f(x(t))) | \leqq \| f \| \bar{E}_x(| F(t) - 1 |) \tag{4.2}$$

where $\| f \| = \sup_{x \in R^N} | f(x) |$.

As $| K(t, w) | \leqq \frac{N}{2} \| \boldsymbol{b} \|^2 t$ for all w, we get

$$\begin{aligned}\bar{E}_x(|F_x(t)-1|)&\leqq \bar{E}_x(e^{|J_x(t)+K(t)|}-1)\leqq \bar{E}_x(e^{N/2\|b\|^2t}e^{|J_x(t)|}-1)\\ &\leqq e^{N/2\|b\|^2t}\sum_{p=0}^{\infty}\frac{\bar{E}_x(|J_x(t)^p|)}{p!}-1\leqq e^{N/2\|b\|^2t}\sum_{p=0}^{\infty}\frac{\sqrt{E_x(J_x(t)^{2p})}}{p!}-1\,. \quad (4.3)\end{aligned}$$

But, by proposition 3.2

$$E(J_x^{2p})\leqq\frac{(2p)!}{p!}(N\|\boldsymbol{b}\|^2 t)^2\leqq p!(4N\|\boldsymbol{b}\|^2 t)^p\,. \quad (4.4)$$

Thus from (4.2), (4.3) and (4.4) we have

$$\begin{aligned}&|E_x(f(x(t))-\bar{E}_x(f(x(t)))|\\ &\leqq\left(e^{N/2\|b\|^2t}-1+\sum_{p=1}^{\infty}e^{N/2\|b\|^2t}\frac{1}{\sqrt{p!}}(4N\|\boldsymbol{b}\|^2t)^{p/2}\right)\|f\|\,. \quad (4.5)\end{aligned}$$

The sum in the right-hand side of (4.5) is uniformly convergent for bounded t.

Therefore, $E_x(f(x(t)))-\bar{E}_x(f(x(t)))$ converges to zero uniformly. q.e.d.

LEMMA 1. *For any $B\in\mathfrak{B}_t$, we have*

$$P_x(B)\leqq C_1\,e^{c_1t}\sqrt{\bar{P}_x(B)} \quad (4.6)$$

$$P(B)\leqq C_2\,e^{c_2t}\sqrt{P_x(B)} \quad (4.7)$$

where C_1, C_2 c_1 and c_2 are positive constants independent of t, x and B.

PROOF.
$$\begin{aligned}P_x(B)&=\bar{E}_x(F_x(t)\chi(B))\leqq\sqrt{\bar{P}_x(B)\bar{E}_x(F_x(t))^2}\\ &\leqq\sqrt{2}\,e^{3N\|b\|^2t}\sqrt{\bar{P}_x(B)} \qquad \text{by theorem 3.4.}\end{aligned}$$

$$\begin{aligned}\bar{P}_x(B)&=E_x\left(\frac{1}{F_x(t)}\chi(B)\right)\leqq\sqrt{P_x(B)E_x\left(\frac{1}{(F_x(t))^2}\right)}\\ &\leqq\sqrt{P_x(B)\bar{E}_x\left(\frac{1}{F(t)}\right)}\leqq 2e^{3/4N\|b\|^2t}\sqrt{P_x(B)} \qquad \text{by theorem 3.4.}\end{aligned}$$

PROPOSITION 4.3.
(i) *If $f\in C$, then $T_tf\in C$,*
(ii) *if $f\in\bar{C}$, then $T_tf\in\bar{C}$,*
(iii) *if $f\in C_\infty$, then $T_tf\in C_\infty$,*
(iv) *if $f\in C_0$, then $T_tf\in C_0$.*

PROOF. For any $w\in W$, we define $w+a$ in W ($a\in R^N$) as

$$x(t,w+a)=x(w,t)+a\equiv(x^i(w,t)+a^i)\,.$$

Then, by the translation invariance of the Brownian motion we have

$$\bar{E}_{x+a}(f(w))=\bar{E}_x(f(w+a)) \tag{4.8}$$

for any $\mathfrak{B}_\infty$-measurable function f (which is integrable with respect to $\bar{P}_x$-measure).

Now,

$$\begin{aligned} I &\equiv T_t f(x+\varepsilon)-T_t(x) \\ &=\bar{E}_{x+\varepsilon}(e^{I_x(t,w)}f(x(t)))-\bar{E}_x(e^{I_x(t,w)}f(x)) \\ &=\lim_{n\uparrow\infty}\{\bar{E}_{x+\varepsilon}(e^{I_n(t,w)}f(x(t)))-\bar{E}_x(e^{I_n(t,w)}f(x(t)))\} \end{aligned}$$

where $I_n(t, w)=J_n(t, w)+K(t, w)$ is defined as in proposition 3.2. Using (4.7) we have

$$\begin{aligned} |I| &= \lim |\bar{E}_x\{e^{I_n(t,w+\varepsilon)}f(x(t)+\varepsilon)-e^{I_n(t,w)}f(x(t))\}| \\ &\leqq \overline{\lim_{n\uparrow\infty}}\,\bar{E}_x(|e^{I_n(t,w+\varepsilon)}-e^{I_n(t,w)}|)\|f\|+\bar{E}_x(e^{I_n(t,w)}|f(x(t)+\varepsilon)-f(x(t))|) \end{aligned}$$

But

$$\begin{aligned} &\overline{\lim_{n\uparrow\infty}}\,\bar{E}_x e^{I_n(t,w)}|f(x(t)+\varepsilon)-f(x(t))| \\ &\leqq \overline{\lim}\sqrt{\bar{E}_x(e^{2I_n(t,w)})\bar{E}_x(f(x(t)+\varepsilon)-f(x(t)))^2} \\ &\leqq 2e^{2N\|b\|^2t}\sqrt{\bar{E}_x(f(x(t)+\varepsilon)-f(x))^2} \qquad \text{by theorem 3.4.} \end{aligned}$$

$\bar{E}_x(f(x(t)+\varepsilon)-f(x))^2$ converges to zero if $f\in C$.* Moreover, if $f\in \bar{C}$. the convergence is uniform with respect to x.

On the other hand, as $\boldsymbol{b}(x)=(b^1(x)\cdots b^N(x))$ is bounded and uniformly continuous, $\int_0^t \frac{1}{2}\sum(b^i(x(t)+\varepsilon)-b^i(x(t)))^2dt$ converges to zero uniformly with respect to n and w. This means by [2] that

$$I_n(t, w+\varepsilon, \boldsymbol{b}(\cdot))=I_n(t, w, \boldsymbol{b}(\cdot+\varepsilon))$$

converges to $I_n(t, w, \boldsymbol{b})$ in $L^2(W, d\bar{P}_x)$-sense uniformly with respect to x and n, and therefore $e^{I_n(t,w+\varepsilon,\boldsymbol{b})}$ converges to $e^{I_n(t,w\boldsymbol{b})}$ in probability $(\bar{P}_x)$ uniformly with respect to n and x.

Then, lemma 1** in section 3 shows the uniform convergence of $\bar{E}_x(|e^{I_n(t,w+\varepsilon)}-e^{I_n(t,w)}|)$ to zero. Thus we have proved (i) and (ii).

* Even if $f\in B$, we can easily prove

$$\bar{E}_x\{f(x|t)+\varepsilon)-f(x|t))\}^2\to 0 \text{ as } \varepsilon\downarrow 0 \text{ for } t>0\,.$$

Thus, the proposition is strengthened as follows: if $f\in B$, then $T_tf\in C$ for $t>0$.

** In this case, instead of one probability measure, the system of probability measures with parameters (namely x) is considered. But the lemma holds also in this case, and we do not need any essential change in the proof.

Now, let D be a compact set in R. Then

$$\lim_{|x|\to\infty} \bar{P}_x(x(t) \in D) = 0$$

and

$$\lim_{|x|\to\infty} P_x(x(t) \in D) = 0 \qquad (4.8) \text{ by lemma 1.}$$

Therefore, if $f \in C_\infty$ and $\lim_{|x|\to\infty} f(x) = a$, then we have

$$|f(x) - a| < \varepsilon, \quad x \notin D_1, \quad \text{for some compact set } D_1,$$

and

$$\begin{aligned} &|E(f(x(t)) - a)| \\ &= |E_x(f(x(t)) - a : x(t) \notin D_1) + E_x(f(x(t)) - a : x(t) \in D_1)| \\ &\leqq |\varepsilon P_x(x(t)) \notin D_1) + (|a| + \|f\|) P_x(x(t) \in D_1) . \end{aligned}$$

Noticing (4.8), we have

$$\lim_{|x|\to\infty} T_t f(x) = \lim_{|x|\to\infty} E(f(x(t)) = a . \qquad (4.9)$$

This result and (i) mean that if $f \in C_\infty$, then $T_t f \in C_\infty$. Especially, if $f \in C_0$, then taking $a = 0$ in (4.9) we have

$$T_t f \in C_0 . \qquad \text{q.e.d.}$$

In spaces C, $\bar{C}$, C_∞ and C_0 the norm is defined as

$$\|f\| = \sup_{x \in R^N} |f(x)| .$$

By means of this norm we can consider those spaces as Banach spaces.

Then, combining propositions 4.1, 4.2 and 4.3, we have the following theorem.

THEOREM 4.4. *The semi-group T_t on B can be restricted on C, $\bar{C}$, C_∞ and C_0. Especially on $\bar{C}$, C_∞ and C_0, this semi-group is strongly continuous.*

REMARK: By the similar argument as in proposition 4.4, we can prove that if $b^i(x)$'s are k-times differentiable and the derivatives of $b^i(x)$'s up to the k-th are contained in $\bar{C}$, then T_t is a mapping from the space of k-times differentiable functions into itself.

5. Representation of the generator

DEFINITION. We define $\bar{\mathfrak{A}}$, $\mathfrak{A}_\infty$ and $\mathfrak{A}_0$ as the generators of the strongly continuous semi-group $\{T_t\}$ on spaces $\bar{C}_0$, C_∞ and C_0 respectively (in the sense of Hille-Yoshida's theorem). $\vartheta(\bar{\mathfrak{A}})$, $\vartheta(\mathfrak{A}_\infty)$ and $\vartheta(\mathfrak{A}_0)$ are their domains.

In this section we shall investigate the explicit local form of these generators and their domains.

PROPOSITION 5.1. *If f is a twice differentiable function, whose derivatives up to the 2nd are continuous and bounded, we have*

$$\lim_{t\to 0}\frac{1}{t}(T_t f(x)-f(x))=Af(x)$$

for any x. If f and their derivatives up to the 2nd are uniformly continuous, then the above convergence is uniform with respect to x.

PROOF. By theorem 1.1 in [2], $e^{I(t,w)}f(x(t, w))$ is a stochastic integral and we get

$$\begin{aligned} de^{I(t)}f(x(t)) = & \Big\{e^{I(t)}f(x(t))\Big(-\sum\frac{b^i(x(t))^2}{2}\Big) \\ & +\frac{1}{2}\Big(e^{I(t)}f(x(t))\Big(\sum\frac{b^i(x(t))^2}{2}\Big)\Big) \\ & +\frac{1}{2}e^{I(t)}\frac{\partial^2 f}{\partial x^{i2}}(x(t))f \mid x(t)+e^{I(t)}\sum\frac{\partial f}{\partial x^i}(x(t))b^i(x(t))\Big\}dt \\ & +e^{I(t)}\sum\Big(b^i(x(t))f(x(t))+\frac{\partial f}{\partial x^i}(x(t))\Big)dx^i(t) \\ = & \, e^{I(t)}Af(x(t))dt+e^{I(t)}(\sum b^i(x(t))+\frac{\partial f}{\partial x^i}(x(t))dx^i(t)) . \end{aligned}$$

Now, $e^{I(t)}\Big(\sum b^i(x(t))+\dfrac{\partial f}{\partial x^i}(x(t))\Big)$ is in $L^2(W\times[0, t], dP_x\times dt)$, so we have by [2]

$$\bar{E}_x(e^{I(t)}f(x(t))-f(x))=\bar{E}_x\Big(\int_0^t e^{I(t)}Af(x(t))dt\Big)=\int_0^t T_tAf(x)dt ,$$

or

$$E_x(f(x(t))-f(x))=\int_0^t T_tAf(x)dt .$$

In proposition 4.2, we have seen that $T_tAf(x)\to Af(x)$ $(t\downarrow 0)$ if $Af\in C$, and the convergence is uniform if $Af\in\bar{C}$. Thus the proposition is proved. q.e.d.

DEFINITION. For $f\in C$, we set

$$S_r f(x)=\frac{N}{r^2}\left\{\int_{\partial K_r}\frac{1}{\omega_N}(f(\xi)+\sum(\xi^i-x^i)b^i(x))dw-f(x)\right\} \tag{5.1}$$

where $K_r\equiv K_r(x)=\{y:\sum|y^i-x^i|^2<r\}$, ω_N is the volume of surface ∂K_r of K_r and $d\omega$ is the volume element of ∂K_r. If $\lim_{r\downarrow 0} S_r f(x)$ exists, we define $Sf(x)$ as

$$Sf(x)=\lim_{r\downarrow 0} S_r f(x) . \tag{5.2}$$

Now, let σ_r be a hitting time to $\partial K_r(x)$, namely,

$$\sigma_r=\inf\{t:x(t)\in\partial K_r\} .$$

Then, σ_r is a Markov time, as $x(t, w)$ is continuous.

DEFINITION. For $f\in C$, we set

$$D_r f(x)=\frac{E_x(f(x(\sigma_r))-f(x))}{E_x(\sigma_r)} . \tag{5.3}$$

When $\lim_{r\downarrow 0} D_r f(x)$ exists, we define $Df(x)$ as

$$Df(x)=\lim_{r\downarrow 0} D_r f(x) . \tag{5.4}$$

Then, by Dynkin's theorem,* we have

PROPOSITION 5.2. (Dynkin)

$\vartheta(\mathfrak{A}_0)=\{f:f\in C_0,\ Df(x)$ *exists for all* x *and* $Df\in C_0\}$

$\vartheta(\mathfrak{A}_\infty)=\{f:f\in C,\ Df(x)$ *exists for all* x *and* $Df\in C_\infty\}$

$\vartheta(\overline{\mathfrak{A}})\subseteqq\{f:f\in C,\ Df(x)$ *exists for all* x *and* $Df\in\bar{C}\}$

and if $f\in\vartheta(\mathfrak{A}_0)$, $\vartheta(\mathfrak{A}_\infty)$ *or* $\vartheta(\overline{\mathfrak{A}})$, *then we have*

$$Df(x)=\mathfrak{A}_0 f(x)\ ,\quad Df(x)=\mathfrak{A}_\infty f(x)\ \text{or}\ Df(x)=\overline{\mathfrak{A}}f(x)$$

for any $x\in R$, *respectively.*

The main result in this section is the following.

PROPOSITION 5.3. *When* $\boldsymbol{b}(x)$ *satisfies Lipschitz condition, namely, for any* $x\in R$ *there exists* L_x *such that*

$$|b^i(y)-b^i(x)|\leqq L_x\sqrt{\sum|y^i-x^i|^2}\ , \tag{5.5}$$

then we have

(i) $$|E_y(f(x(\sigma_r)))-\bar{E}_y(f(x(\sigma_r))|<Cr \tag{5.6}$$

for any $y\in K_r(x)$ *where* C *is independent of* x *and* y *and*

(ii) $$\lim_{r\downarrow 0}|D_r f(x)-S_r f(x)|=0 . \tag{5.7}$$

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* c.f foot not in page 56.

The second statement (5.7) shows that $Sf(x)$ exists if and only if $Df(x)$ exists, and in this case $Sf(x)=Df(x)$. To prove the proposition we need several lemmas.

LEMMA 1.
$$\bar{E}_x(\sigma_r)=\frac{1}{N}r^2\,, \tag{5.8}$$
$$\bar{E}_x(\sigma_r^n)=k_n r^{2n} \tag{5.9}$$
where k_n is a constant, independent of r and x.

The proof is easily obtained by properties of the Brownian motion. Especially (5.9) is equivalent to the existence of the nth moment of $\sigma_1=\sigma_{\partial K_1}$.

LEMMA 2.
$$\bar{E}_x(x^i(\sigma_r)-x^i)(x^j(\sigma_r)-x^j)=\delta^{ij}\frac{r^2}{N}\,. \tag{5.10}$$

LEMMA 3. *Let σ be a bounded Markov time ($\sigma<T$), and*
$$J_x(\sigma,\boldsymbol{b})=\int_0^T\chi(\sigma\geqq s)\sum b^i(x(s))dx^i(s)\,. \tag{5.11}$$
Then, we have
$$\bar{E}_x(J(\sigma,\boldsymbol{b})^2)\leqq C_1\,||\,\boldsymbol{b}\,||^2\,\bar{E}_x(\sigma)$$
$$\bar{E}_x(J(\sigma,\boldsymbol{b})^{2n})\leqq C_n\sqrt{T}\,||\,\boldsymbol{b}\,||^{2n}\sqrt{\bar{E}_x(\sigma^{2n-1})}\quad n\geqq 2 \tag{5.12}$$
where C_n's are independent of $\boldsymbol{b}$ and x and $||\,\boldsymbol{b}\,||=\sup_{i,x}|\,b^i(x)\,|$.

PROOF. First, we remark that $J_x(\sigma,\boldsymbol{b})$ is well defined as a stochastic integral, for $\chi(\sigma<s)\sum b^i(x(s))\in\mathfrak{B}_s$. Set
$$J_t(\sigma)=J_t(\sigma,\boldsymbol{b})=\int_0^t\chi(\sigma\geqq s)\sum b^i(x(s))dx^i(s)\,.$$
Then $J_T(\sigma)\equiv J(\sigma,\boldsymbol{b})$, $J_0(\sigma)=0$ and $J_t(\sigma)^{2n}$ is a stochastic interal which can be written as
$$\begin{aligned}dJ_t(\sigma)^{2n}=&2nJ_t(\sigma)^{2n-1}\chi(t\leqq\sigma)\sum b^i(x(t))dx^i(t)\\&+2n(2n-1)J_t(\sigma)^{2n-2}\sum b^i(x(t))^2\chi(t\leqq\sigma)dt\,.\end{aligned}$$
For almost all w such as $t\leqq\sigma(w)$ and $u<t$, we get
$$\chi(u\leqq\sigma)b^i(x(u))=b^i(x(u))\,.$$
Therefore, $J_t(\sigma)=J(t)$ for almost all w with $t\leqq\sigma(w)$ by [2]. Thus,

$$dJ_t(\sigma)^{2n}=2nJ(t)^{2n-1}\chi(t\leqq\sigma)\sum b^i(x(t))dx^i(t)$$
$$+2n(2n-1)J(t)^{2n-2}\sum b^i(x(t))^2\chi(t\leqq\sigma)dt\ .$$

According to proposition 3.2 we have

$$2E\int_0^t(2nJ(t)^{2n-1}\chi(t\leqq\sigma))^2\sum b_i^2(x(t))dt<\infty\ ,$$

and

$$\bar{E}_x(J(\sigma)^{2n})=\bar{E}_x(J_T(\sigma)^{2n})=\int_0^T 2n(2n-1)\bar{E}_x\{J(s)^{2n-2}\chi(\sigma\geqq s)\sum b^i(x(s))^2\}ds\ .$$

If $n=1$,

$$\bar{E}_x(J(\sigma)^2)\leqq 2N\|\boldsymbol{b}\|^2\int_0^T\bar{P}_x(\sigma\geqq s)ds=2N\|\boldsymbol{b}\|^2\bar{E}_x(\sigma)\ .$$

If $n\geqq 2$, according to the relation

$$\bar{E}_x(J(s)^{4n-4})\leqq c_n'\|\boldsymbol{b}\|^{4n-4}s^{2n-2}$$

in proposition 3.2, we have

$$\begin{aligned}\bar{E}_x(J(\sigma)^{2n})&\leqq 2n(2n-1)N\|\boldsymbol{b}\|^2\int_0^T\sqrt{\bar{E}_x(J(s)^{4n-4})\bar{P}_x(\sigma\geqq s)}\,ds\\&\leqq c_n''\|\boldsymbol{b}\|^{2n}\int_0^T\sqrt{s^{2n-2}\bar{P}_x(\sigma\geqq s)}\,ds\\&\leqq c_n''\|\boldsymbol{b}\|^{2n}\sqrt{T}\sqrt{\int_0^T\bar{P}_x(\sigma\geqq s)s^{2n-2}\,ds}\\&\leqq c_n\sqrt{T}\|\boldsymbol{b}\|^{2n}\sqrt{\bar{E}_x(\sigma^{2n-1})}\ .\end{aligned}$$

q.e.d.

LEMMA 4. *Let* $\sigma_r=\sigma_{\partial K_r(x)}$, $\sigma_T=\min(\sigma_r, T)$, *where* T *is a positive constant and*

$$I_x(\sigma_T, \boldsymbol{b})=J(\sigma_T, \boldsymbol{b})+\int_0^{\sigma_T}\left(-\frac{1}{2}\sum b^i(x(t))^2\right)dt\ . \tag{5.13}$$

Then, we have

$$\bar{E}_x(I(\sigma_T, \boldsymbol{b})^2)\leqq C_1(\|\boldsymbol{b}\|^2r^2+\|\boldsymbol{b}\|^4r^4)$$
$$\bar{E}_x(I(\sigma_T, \boldsymbol{b})^{2n})\leqq C_n(\sqrt{T}\|\boldsymbol{b}\|^{2n}r^{2n-1}+\|\boldsymbol{b}\|^{4n}r^{4n})\qquad n\geqq 2\ ,$$

where C_n*'s are positive constants independent of* r, x *and* $\boldsymbol{b}$.

PROOF. Using lemma 3, we have

$$\bar{E}_x(I(\sigma_T, \boldsymbol{b})^{2n})\leqq 2^{2n}\left\{\bar{E}_x(J(\sigma_T, \boldsymbol{b})^{2n})+\bar{E}_x\left(\int_0^{\sigma_T}\frac{1}{2}\sum b^i(x(t))^2dt\right)^{2n}\right\}$$

$$\leqq C_1'(\| \boldsymbol{b} \|^2 \bar{E}_x(\sigma_T) + \| \boldsymbol{b} \|^4 \bar{E}_x(\sigma_T^2)) \qquad \text{if } n=1 ,$$

$$\leqq C_n'(\sqrt{T} \| \boldsymbol{b} \|^{2n} \sqrt{\bar{E}_x(\sigma_T^{2n-1})} + \| \boldsymbol{b} \|^{4n} \bar{E}_x(\sigma_T^{2n})) \quad \text{if } n \geqq 2 .$$

By lemma 1, we have

$$\bar{E}_x(\sigma_T^n) \leqq \bar{E}_x(\sigma_r^n) \leqq k_n r^{2n} ,$$

and

$$\bar{E}_x(I(\sigma_T, \boldsymbol{b})^2) \leqq C_1(\| b \|^2 r^2 + \| b \|^4 r^4) \qquad \text{if n=2} ,$$

$$\bar{E}_x(I(\sigma_T, \boldsymbol{b})^{2n}) \leqq C_n(\sqrt{T} \| b \|^{2n} r^{2n-1} + \| b \|^{4n} r^{4n}) \text{ if } n \geqq 2 . \text{ q.e.d.}$$

Let σ_k $(k=0, 1, 2, \cdots, m)$ be a Markov time such as $0=\sigma_0 \leqq \sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_m \leqq T$ where T is a fixed constant, and let $\boldsymbol{f}_k(w)=(f_k^1(w), \cdots, f_k^N(w))$ $(k=0, 1, 2, \cdots, m)$ be $\mathfrak{B}_{\sigma_k+}$-measurable vector random variables which satisfy $\sup\limits_{k,i,w} | f_k^i(w) | < M < \infty$. Define

$$\boldsymbol{f}(t) = \begin{cases} \boldsymbol{f}_0(w) & \text{if } t=0 , \\ \boldsymbol{f}_k(w) & \text{if } \sigma_k < t \leqq \sigma_{k+1} \ k=0, \cdots, m-1 , \\ 0 & \text{if } \sigma_m < t . \end{cases} \tag{5.14}$$

Then $\boldsymbol{f}(t) = \sum \chi(\sigma_k < t) \chi(t \leqq \sigma_{k+1}) \boldsymbol{f}_k(w)$ is $\mathfrak{B}_t$ measurable. Therefore, the stochastic integral $\int_0^t \sum f^i(t) dx^i(t)$ is well defined.

LEMMA 5. *We have*

$$\int_0^T \sum_{i=1}^N f^i(t)\, dx^i(t) = \sum_{k=1}^n \Big(\sum_{i=1}^n f_k^i(w)(x^i(\sigma_{k+1}) - x^i(\sigma_k)) \Big) \tag{5.15}$$

almost surely with respect to $\bar{P}_x$-measure.

PROOF. Let $0=t_0<t_1< \cdots <t_n=T$. If $\sup\limits_l | t_{l+1} - t_l | \to 0$, then a suitable subsequence of $\{I_n\} = \Big\{ \sum\limits_{l=1}^{n-1} \sum\limits_i f^i(t_l)(x^i(t_{l+1}) - x^i(t_l)) \Big\}$ converges almost surely to the stochastic integral

$$\int_0^T \sum f^i(t) dx^i(t) .$$

On the other hand, since $\boldsymbol{f}(t)=0$ for $t>\sigma_m$, we have

$$\sum_{k=0}^{m-1} \sum_i f_k^i(w)(x^i(\sigma_{k+1}) - x^i(\sigma_k)) - I_n$$

$$= \sum_{k=0}^{m-1} \sum_i \Big\{ f_k^i(w)(x^i(\sigma_{k+1}) - x^i(\sigma_k)) - \sum_{l=I(k)}^{I(k+1)-1} f^i(t_l)(x^i(t_{l+1}) - x^i(t_l)) \Big\} \tag{5.16}$$

where $I(k)$'s are the random variables such that

$$t_{I(k)-1} \leqq \sigma_k < t_{I(k)} < t_{I(k)+1} < \cdots < t_{I(k+1)-1} \leqq \sigma_{k+1} < t_{I(k+1)} < \cdots .$$

Then, as $f(t_l) = f_k(w)$ for $I(k) \leqq l < I(k+1)$, the above difference (5.16) is equal to

$$\sum_{k=0}^{m-1} \sum f_k^i(w)\{(x^i(\sigma_{k+1}) - x^i(t_{I(k+1)})) - (x^i(\sigma_k) - x^i(t_{I(k)}))\} . \tag{5.17}$$

From the continuity of $x(t)$ process, (5.17) is seen to converge to zero (for all ω) if $\sup|t_{l+1} - t_l|$ converges to zero. q.e.d.

LEMMA 6. *Let $f(x, w)$ be a function on $R^1 \times W$ which is $\mathfrak{B}^1 \times \mathfrak{B}_\infty$-measurable and $X(w)$ is a $\mathfrak{B}_{\sigma+}$-measurable real random variable. Then*

$$\bar{E}_x(f(X(w), w_\sigma^+) \mid \mathfrak{B}_{\sigma+}) = \bar{E}_{x(\sigma)}(f(x, w))\,|_{x=X(w)} \tag{5.18}$$

almost surely (with respect to $\bar{P}_x$-measure).

PROOF. First, we notice that (5.18) is easily proved for the case where $f(t, w) = g(x)b(w)$. Then, from this it is seen to follow that (5.18) holds in general.

LEMMA 7. *Let $f(w)$ be a bounded $\mathfrak{B}_\sigma$-measurable function, where σ is a bounded Markov time ($\sigma < T$). Then, we have*

$$E_x(f(w)) = \bar{E}_x(e^{I_x(\sigma, \boldsymbol{b})} f(w)) \tag{5.19}$$

where $I_x(\sigma, \boldsymbol{b})$ is defined in the same way as in lemma 4.

PROOF. As $\sigma(w) < T$, we have $\mathfrak{B}_\sigma \subset \mathfrak{B}_T$, and it is sufficient to prove that

$$\bar{E}_x(e^{I_x(T, \boldsymbol{b})} f(w)) = \bar{E}_x(e^{I_x(\sigma, \boldsymbol{b})} f(w)) . \tag{5.20}$$

Let $0 = t_0 < t_1 < \cdots < t_n = T$, and define

$$\begin{aligned} \tilde{\boldsymbol{b}}_n(s) &= \boldsymbol{b}(x(t_k))\chi(t_k \leqq \sigma) && \text{if } t_k \leqq s < t_{k+1} \\ \tilde{\tilde{\boldsymbol{b}}}_n(s) &= 0 && \text{if } s \leqq \sigma \\ &= \boldsymbol{b}(x(\sigma + t_k))\chi(\sigma + t_k < T) && \text{if } \sigma + t_k < s \leqq \sigma + t_{k+1} . \end{aligned}$$

Then, we can define stochastic integrals

$$\tilde{J}_n = \int_0^T \sum \tilde{b}_n^i(s) dx^i(s) ,$$

$$\tilde{\tilde{J}}_n = \int_0^T \sum \tilde{\tilde{b}}_n^i(s) dx^i(s)$$

and

$$\tilde{I}_n=\tilde{J}_n+\int_0^\sigma\Big(-\frac{1}{2}\sum b^i(x(s))^2\Big)ds\ ,$$

$$\tilde{\tilde{I}}_n=\tilde{\tilde{J}}_n+\int_\sigma^T\Big(-\frac{1}{2}(\sum b^i(x(s))^2\Big)ds\ .$$

Applying lemma 5 to $\tilde{\tilde{J}}_n$, we have

$$\begin{aligned}\tilde{\tilde{J}}_n&=\sum_{k=1}^{n-1}\chi(\sigma+t_k<T)\sum_i b^i(x(\sigma+t_k))(x^i(\sigma+t_{k+1})-x^i(\sigma+t_k))\\&=\sum_{k=0}^{n-1}\chi(\sigma+t_k<T)\sum_i b^i(x(t_k,\ w_\sigma^+))(x^i(t_{k+1},\ w_\sigma^+)-x(t_k,\ w_\sigma^+))\ ,\end{aligned}$$

and

$$\int_\sigma^T\Big(-\frac{1}{2}\sum b^i(x(s))^2\Big)ds=\int_0^{T-\sigma}\Big(-\frac{1}{2}\sum b^i(x(s,\ w_\sigma^+))^2\Big)ds\ .$$

Thus, we have

$$\bar{E}_x(e^{\tilde{I}_n+\tilde{\tilde{I}}_n}f(w))=E\{e^{\tilde{I}_n}f(w)\bar{E}_x(e^{\tilde{\tilde{I}}_n}\mid\mathfrak{B}_{\sigma+})\}\ . \tag{5.21}$$

On the other hand, by lemma 6, we have

$$\bar{E}_x(e^{\tilde{\tilde{I}}_n}\Big|\ \mathfrak{B}_{\sigma+})=\bar{E}_{x(\sigma)}\Big(e^{\sum_{k=0}^{n-1}\chi(t_k\leq T-\rho)\sum_i b^i(x(t_k))(x^i(t_{k+1})-x^i(t_k))-\frac{1}{2}\int^{T-\rho}\sum b^i(x(s))^2ds}\Big)\Big|_{\rho=\sigma}\ .$$

therefore,

$$\bar{E}_x(e^{\tilde{I}_n+\tilde{\tilde{I}}_n}f(w))=\bar{E}_x(e^{\tilde{I}_n}f(w)\bar{E}_{x(\sigma)}(e^{I_n(T-\rho)})|_{\rho=\sigma})\ .$$

Now, if $\sup_k|t_{k-1}-t_k|$ converges to zero, then

$$\int_0^T\sum(\tilde{b}_n^i(s)+\tilde{\tilde{b}}_n^i(s)-b^i(x(s)))^2ds\ \text{ and }\ \int_0^T\sum(\tilde{b}^i(s)-b^i(x(s))\chi(s\leqq\sigma))^2ds$$

converge to zero. Hence by [2] $\tilde{I}_n+\tilde{\tilde{I}}_n$, $\tilde{I}_n$ and $I_n(t-\rho)$ converge to $I_x(T)$, $I_x(\sigma)$ and $I_x(t-\rho)$ in L^2-sense, respectively.* Moreover, since $\bar{E}_x(e^{\tilde{I}_n+\tilde{\tilde{I}}_n}f(w))^2$, $\bar{E}_x(e^{\tilde{I}_n}f(w)\bar{E}_{x(\sigma)}(e^{I_n(t-\rho)})|_{\rho=\sigma})^2$ and $\bar{E}_{x(\sigma)}(e^{I_n(t-\rho)})^2$ are uniformly bounded as is mentioned in the remark to theorem (3.4), we can apply lemma 1 in section 3 and get

$$\bar{E}_x(e^{I(T)}f(w))=\bar{E}_x(e^{I(\sigma)}f(w)\bar{E}_{x(\sigma)}(e^{I(T-\rho)})|_{\rho=\sigma})\ .$$

By (3.13) in proposition 3.6, we have

$$\bar{E}_{x(\sigma)}(e^{I(T-\rho)})=P_{x(\sigma)}(W)\equiv 1\ .$$

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* Especially, the L^2-convergence of $I_n(t-\rho)\to I_x(t-\rho)$ is uniform with respect to x and ρ,

Thus, (5.20) is proved. q.e.d.

LEMMA 8. *$E_y(\sigma_r)<cr^2$ for $r<1/(\|b\|)$ and $y\in K_r(x)$ where c is a positive constant independent of r.*

PROOF. We can construct a twice differentiable function f whose derivatives up to the 2nd are in C_0 and which satisfies the relation:

$$f(y)=\frac{2}{N}\sum(y^i-x^i)^2 \qquad \text{for any } y\in K_r(x)$$

and $\|f\|=\sup|f(x)|\leqq cr^2$ in the whole space. Then, by proposition 5.1, $f\in\vartheta(\mathfrak{A}_0)\subset\vartheta(\mathfrak{A}_\infty)\subset\vartheta(\overline{\mathfrak{A}})$ and

$$Af(y)=1+\sum\frac{4}{N}b^i(y)(y^i-x^i) \qquad \text{for } y\in K_r$$

or

$$Af(y)>1-4\|b\|r\ .$$

By Dynkin's lemma* we get

$$\begin{aligned}E_x(\sigma_r)(1-4\|b\|r) &\leqq E_x\Big(\int_0^{\sigma_r}Af(x(s))ds\Big)\\ &=E_x(f(x(\sigma_r)))-f(x)\leqq 2\|f\|\leqq 2c_1r^2\ .\end{aligned}$$

Now, as $1-4\|b\|r>\frac{1}{2}$ by the assumption, we have

$$E_x(\sigma_r)\leqq cr^2\ . \qquad \text{q.e.d.}$$

PROOF OF THE PROPOSITION 5.3.

Setting $\sigma=\sigma_r$ and $\bar\sigma=\min(\sigma, T)$, where T is a fixed positive constant, we have

$$\begin{aligned}I_1=|E_y(f(x(\sigma)))-E_y(f(x(\sigma))\chi(\sigma<T))| &\leqq \|f\|\,\bar E_y(e^{I(T)}\chi(\sigma\geqq T))\\ &\leqq \|f\|\sqrt{\bar E_y(e^{2I(T)})\bar P_y(\sigma\geqq T)}\end{aligned}$$

for any $y\in K_r(x)\equiv K_r$. As $\sigma^*=\sigma_{\partial K_{2r(y)}}\geqq\sigma_{\partial K_{r(x)}}$, we have

$$\bar E_y(\sigma^n)\leqq\bar E_y(\sigma^{*n})\leqq kr^{2n}$$

by lemma 1, and then $\bar P_y(\sigma\geqq T)\leqq cr^{2n}$, where c is independent of x, y and r. Therefore

$$I_1\leqq c_1r^3\ . \tag{5.22}$$

* The fact that $(P_x, W\mathfrak{B}_\infty)$-process has a strong Markov property, is easily deduced if we notice that the strongly continuous semigroup $\{T_t\}$ can be restricted on C_∞ and every element $x(t, w)$ in W is continuous with respect to t. We can also prove this fact directly, modifying the proof of lemma 7.

As

$$E_y(f(x(\sigma))\chi(\sigma<T))=E_y(f(x(\bar\sigma))\chi(\bar\sigma<T))$$
$$=E_y(e^{I(\bar\sigma)}f(x\bar\sigma))\chi(\bar\sigma<T)$$

by lemma 7, we have

$$\begin{aligned}I_2&=|\,E_y(f(x(\bar\sigma))\chi(\bar\sigma<T))-\bar E_y(f(x(\sigma))\chi(\sigma<T))|\\&\leqq \bar E_y(|\,e^{I(\bar\sigma)}-1\,|)||f||\leqq \bar E_y(e^{|I(\sigma)|}|\,I(\bar\sigma)|)||f||\\&\leqq ||f||\sqrt{\bar E_y(I(\bar\sigma)^2)E(e^{2I(\bar\sigma)}+e^{-2I(\bar\sigma)})}\,.\end{aligned}$$

Then, by lemma 4 and the remark to theorem 3.4, we have

$$I_2\leqq c_2 r\,. \tag{5.23}$$

The same argument as in proving (5.22) leads to

$$I_3=|\,\bar E_y(f(x(\sigma))\chi(\sigma<T))-\bar E_y(f(x(\sigma)))|<c_3r^3\,. \tag{5.24}$$

When noticing that c_1, c_2 and c_3 are independent of x, y and r by (5.22), (5,23) and (5,24), we have proposition 5.3 (i), (5.6). Now, taking $y=x$ in (1), we have

$$I_1'=|\,E_x(f(x(\sigma)))-\bar E_x(f(x(\sigma))\chi(\sigma<T))|<c_1r^3 \tag*{(5.22)$'$.}$$

and

$$\begin{aligned}I_4&=\left|E_x(f(x(\bar\sigma))\chi(\bar\sigma<T))-\bar E_x\Big\{f(x(\bar\sigma))\Big(1+I(\bar\sigma)+\frac{1}{2}I(\bar\sigma)^2\Big)\chi(\bar\sigma<T)\Big\}\right|\\&\leqq\frac{||f||}{3}\bar E_x(|\,I(\bar\sigma)|^3\,e^{I(\bar\sigma)})\leqq\frac{||f||}{3}\sqrt{\bar E_x(I(\bar\sigma)^6)\bar E_x(e^{2I(\bar\sigma)}+e^{-2I(\bar\sigma)})}\,.\end{aligned}$$

Then, by lemma 4 and the remark to theorem 3.4, we have

$$I_4\leqq c_4r^{5/2}\,. \tag*{(5.25)*}$$

Consider

$$\begin{aligned}I_5=\Big|\bar E_x&\Big\{\Big(1-I(\bar\sigma)+\frac{1}{2}I(\bar\sigma)^2\Big)\chi(\bar\sigma<T)f(x(\bar\sigma))\Big\}\\&-\bar E_x\Big\{\Big(1+I_0(\bar\sigma)+\frac{1}{2}I_0(\bar\sigma)^2\Big)f(x(\bar\sigma))\chi(\bar\sigma<T)\Big\}\Big|\end{aligned}$$

where

$$I_0(\bar\sigma)=\int_0^T\chi(\sigma\geqq s)\sum b^i(x)dx^i(s)-\frac{1}{2}\int_0^{\bar\sigma}\sum_i b^i(x)^2ds$$

* Without loss of generality we can assume that $r<1$.

$$= \sum_i b^i(x)(x(\bar{\sigma})-x) - \frac{1}{2}(\sum_i b^i(x)^2)\bar{\sigma} \qquad \text{(c.f. lemma 5).}$$

Taking into account the Lipschitz condition (5.5) and

$$\varepsilon(r) = \sup_{y \in K_r(x)} |f(y)-f(x)| \downarrow 0 \ (r \downarrow 0)\ , \tag{5.26}$$

we have

$$|\bar{E}_x(I(\bar{\sigma})-I_0(\bar{\sigma}))f(x)| \leqq \| f \| \bar{E}_x\left\{\int_0^\sigma \sum |b^i(x(s))^2 - b^i(x)^2|\, ds\right\}$$
$$\leqq 2 \| f \| \, \| b \| N L_x r \bar{E}_x(\bar{\sigma}) \leqq c' r^3$$

$$\bar{E}_x\{| I(\bar{\sigma})-I_0(\bar{\sigma})| \, |f(x(\bar{\sigma}))-f(x)|\} \leqq \varepsilon(r)\sqrt{\bar{E}_x(I(\bar{\sigma})-I_0(\bar{\sigma}))^2}$$
$$= \varepsilon(r)\sqrt{\bar{E}_x(I(\bar{\sigma}, b(\cdot)-b(x))^2)}$$
$$\leqq c'' \varepsilon(r) r^2 \qquad \text{(by lemma 4),}$$

and

$$\bar{E}_x\left\{\frac{1}{2}|(I(\bar{\sigma})-I_0(\bar{\sigma}))(I(\bar{\sigma})+I_0(\bar{\sigma}))f(x(\bar{\sigma}))|\right\}$$
$$\leqq \bar{E}_x\left\{\frac{1}{2}| I(\bar{\sigma}, b(\cdot)-b(x))(I(\bar{\sigma})+I_0(\bar{\sigma}))|\right\} \| f \|$$
$$\leqq \frac{1}{2}\sqrt{\bar{E}_x(I(\bar{\sigma}, b(\cdot)-b(x))^2)\bar{E}_x(I(\bar{\sigma})+I_0(\bar{\sigma}))^2}\, \| f \|$$
$$\leqq c''' r^3 \qquad \text{(by lemma 4),}$$

thus,

$$I_5 \leqq |\bar{E}_x\{(I(\bar{\sigma})-I_0(\bar{\sigma}))f(x)\}| + \bar{E}_x\{| I(\bar{\sigma})-I_0(\bar{\sigma})| \, |f(x(\bar{\sigma}))-f(x)|\}$$
$$+ \frac{1}{2}\bar{E}_x\, |(I(\bar{\sigma})-I_0(\bar{\sigma}))(I(\bar{\sigma})+I_0(\bar{\sigma}))| \leqq c_5 r^2(\varepsilon(r)+r)\ . \tag{5.27}$$

By the same procedure as in proving (5.22), we can prove

$$I_6 \leqq \left| \bar{E}_x\left\{\left(1+I_0(\bar{\sigma})+\frac{1}{2}I_0(\bar{\sigma})^2\right)f(x(\bar{\sigma}))\chi(\bar{\sigma}<T)\right\}\right.$$
$$\left. - \bar{E}_x\left\{\left(1+I_0(\sigma)+\frac{1}{2}I_0(\sigma)\right)f(x(\sigma))\right\}\right| \leqq c_6 r^3\ , \tag{5.28}$$

where $I_0(\sigma) = \sum_i b(x(\sigma))(x^i(\sigma)-x) - \frac{1}{2}(\sum b^i(x)^2)\sigma$. Finally, we get

$$I_7 = \left| \bar{E}_x\left\{\left(1+I_0(\sigma)+\frac{1}{2}I_0(\sigma)^2\right)f(x(\sigma))\right\} - \bar{E}_x\{(1+\sum b^i(x)(x^i(\sigma)-x^i)f(x(\sigma))\}\right|$$

$$\leqq \bar{E}_x\Big\{\Big(-\frac{1}{2}(\sum b^i(x)^2)\sigma+\frac{1}{2}(\sum b^i(x)(x^i(\sigma)-x^i))^2\Big)f(x)\Big\}$$

$$+\bar{E}_x\Big\{\Big|\Big(-\frac{1}{2}(\sum b^i(x)^2)\sigma+\frac{1}{2}(\sum b^i(x)(x^i(\sigma)-x^i))^2\Big)(f(x(\sigma)-f(x)))\Big|\Big\}$$

$$+\bar{E}_x\Big\{\frac{1}{4}(\sum b^i(x)^2)^2\sigma^2+|(\sum b^i(x(\sigma))(x^i(\sigma)-x^i))|\Big(\sum b^i(x)^2\sigma\Big)\Big\}\|f\|\,.$$

The first term vanishes by lemmas 1 and 2, the second term is less than $c\varepsilon(r)r^2$ by lemma 1, (5.26) and the fact that $\sum(x^i(\sigma)-x^i)^2=r^2$, and the third term is less than cr^3 by lemma 1. Thus we have

$$I_7\leqq c_7r^2(\varepsilon(r)+r) \tag{5.29}$$

By (5.22), (5.25), (5.27), (5.28) and (5.29), we have

$$E_x(f(x(\sigma)))-\bar{E}_x(1+\sum b^i(x)(x^i(\sigma)-x^i))f(x(\sigma))\leqq r^2\varepsilon^*(r) \tag{5.30}$$

where $$\varepsilon^*(r)\downarrow 0 \qquad \text{if } r\downarrow 0\,.$$

On the other hand,

$$0\leqq E_x(\sigma)-E_x(\bar{\sigma})\leqq E_x((\sigma-T)\chi(\sigma\geqq T))\,,$$

$$E_x((\sigma-T)\chi(\sigma\geqq T))=E_x(\chi(\sigma\geqq T)E_{x(T)}(\sigma))\leqq P_x(\sigma\geqq T)cr^2 \quad \text{by lemma 8,}$$

and $P_x(\sigma\geqq T)\leqq cr^2$ by the same lemma. Hence we have

$$|E_x(\sigma)-E_x(\bar{\sigma})|\leqq cr^4 \tag{5.31}$$

Similary,

$$|\bar{E}_x(\sigma)-\bar{E}_x(\bar{\sigma})|\leqq cr^4\,. \tag{5.32}$$

$$|E_x(\bar{\sigma})-\bar{E}_x(\bar{\sigma})|\leqq \bar{E}_x(|e^{I(\bar{\sigma})}-1|\,\bar{\sigma})|\leqq \sqrt{\bar{E}_x(I(\bar{\sigma}e^{2I(\bar{\sigma})})\bar{E}_x(\bar{\sigma}^2)}\,.$$

By the same argument as in proving (5.23) (therefore, by lemma 4 and the remark to theorem 3.2) we get

$$|E_x(\bar{\sigma})-\bar{E}_x(\bar{\sigma})|\leqq cr^{5/2}\,. \tag{5.33}$$

By (5.31), (5.32) and (5.33), we have

$$|E_x(\sigma)-\bar{E}_x(\sigma)|\leqq cr^{5/2}$$

or $$\Big|E_x(\sigma)-\frac{r^2}{N}\Big|<cr^{5/2} \qquad \text{(5.34) by lemma 1.}$$

Since $x(\sigma)$ is uniformly distributed on $\partial K_r(x)$ by $\bar{P}_x$-measure, it is seen by (5.30) and (5.34), that (5.7) (proposition 5.3 (ii)) holds true.

Thus, all the statements of proposition 5.3 have been proved.

Combining propositions 5.2 and 5.3, we have the following

THEOREM 5.4. *If $b(x)$ satisfies the Lipschitz condition* (5.5), *then*

(i) $S \equiv D$ *(this also means that the domains of S and D coincide with each other for any $x \in R^N$).*

(ii) $\vartheta(\mathfrak{A}_0)=C \cap \{f : Sf \text{ exists and } Sf \in C_\infty\}$,
$\vartheta(\mathfrak{A}_\infty)=C_\infty \cap \{f : Sf \text{ exists and } Sf \in C_0\}$,
$\vartheta(\overline{\mathfrak{A}}) \subset \bar{C} \cap \{f : Sf \text{ exists and } Sf \in \bar{C}\}$,

and

$$\text{If } f \in \vartheta(\mathfrak{A}_0),\ \vartheta(\mathfrak{A}_\infty) \text{ or } \vartheta(\overline{\mathfrak{A}}), \text{ then } Sf \equiv \mathfrak{A}_0 f,\ \mathfrak{A}_\infty f \text{ or } \overline{\mathfrak{A}} f.$$

6. Remark to the properties of the process

As $\bar{P}_x$-measure and P_x-measure are absolutely continuous with respect to each other if they are restricted on $\mathfrak{B}_t$, a condition which is determined within a finite time is satisfied almost surely for $(P_x, W, \mathfrak{B}_\infty)$-process if and only if it is satisfied for the Brownian motion.

For example:

1. For $N=1$, the law of the local iterated logarithm is true for $(P_x, W, \mathfrak{B}_\infty)$-process. For $N \geqq 2$, the similar law holds, too.

2. If F is a compact set in R, then $P_x(x(t) \in F$ for some $t>0)>0$ if and only if the capacity of F is positive.

PROOF. $P_x(x(t) \in F$ for some $t)=0$
if and only if $P_x(x(t) \in F$ for some $t<T)=0$ for any T.
By proposition 3.1, we have

$$P_x(x(t) \in F \text{ for some } t<T)=0$$

if and only if $\bar{P}_x(x(t) \in F$ for some $t<T)=0$
or equivalently $\bar{P}_x(x(t) \in F$ for some $t>0)=0$. The assertion is well known for the Brownian motion ([5]). q.e.d.

3. For $N=2$ or 3, the path of the process has infinitely many double points almost surely. On the other hand, for $N \geqq 4$ the process has no double point almost surely (c.f [5]).

Etc.

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