ON THE STATISTICAL CONTROL OF THE GAP PROCESS

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0. Introduction and summary

In the former paper [1] the present author reported a result of the analysis of statistical structure of the automobile flow in the street. There we have observed a remarkable agreement of the observed data with a model of recurrent type zero-one process and the possibility of application of the model to the reeling process was suggested. Since that time applications of the model which we call the gap process in this paper was developed by Shimazaki at the Sericultural Experiment Station of the Ministry of Agriculture and Forestry to construct the statistical system for controlling the reeling process with fixed number of cocoons. Part of his results were presented in the papers [9, 10] and a discussion of his results was made by the present author [2]. In the course of the development of the applications it was then recognized that the variance-time curve which will be defined in § 2 of this paper plays central role in the construction of the system. And the discussion of its sampling fluctuations has become necessary to settle the theoretical foundation of the system.

The purpose of this paper is to present the asymptotic distribution of an estimate of the variance-time curve and to discuss the construction of a statistical control system of the gap process.

Main analytical tool used in this paper is the method of differentiable statistical function developed by Mises in his paper [7] combined with the generating function method. The results show that our estimate will be practically applicable to most of the practical cases using a sample of moderate size of the primary gaps.

Shimazaki has recently succeeded in establishing an overall statistical control system for the reeling process and the results of this paper were found to be sufficient as the foundation of his system [11].

In § 1 of this paper we shall give the definition of the gap process and describe some of its foundamental properties. In § 2 an estimation procedure to get the variance-time curve of the gap process is given.

Asymptotic distribution of the estimate is obtained with the aid of the differentiable statistical function. In § 3 numerical examples are given to show the practical applicability of the asymptotic results. We discuss in § 4 the construction of a control system of the gap process under the typical condition suggested by the reeling process. The discussion is made on the basis of the variance-time curve of the process and an interpretation of the results obtained by using the differentiable statistical function is given. Some comments are made in § 5 on the practical application of our method. These comments will be of importance to those who want to use the gap process as an approximation to some continuous time parameter process.

1. Definition of the gap process

We shall here define the gap process $\{X_n(\omega); -\infty<n<\infty\}$ with the gap distribution $\{p_\nu; \nu=1, 2, \cdots\}$ as a strictly stationary process of discrete time parameter with the finite dimensional distributions satisfying

$$\mathrm{Prob}\{(X_{n+1}(\omega), X_{n+2}(\omega), \cdots X_{n+k}(\omega))=(\alpha_1, \alpha_2, \cdots \alpha_k)\}$$
$$=P\sum_{\mu>k}\sum_{\nu\geq\mu} p_\nu \quad \text{when all the } \alpha_i\text{'s are equal to } 0,$$
$$=P\Big(\sum_{\nu\geq\varepsilon_1} p_\nu\Big)p_{\varepsilon_2-\varepsilon_1}p_{\varepsilon_3-\varepsilon_2}\cdots p_{\varepsilon_r-\varepsilon_{r-1}}\Big(\sum_{\mu>k-\varepsilon_r} p_\mu\Big)$$

when $\alpha_{\varepsilon_1}, \alpha_{\varepsilon_2}\cdots\alpha_{\varepsilon_r}$ arranged in ascending order of suffixes are equal to 1 and other α_j's are equal to zero,

where $\{p_\nu; \nu=1, 2, \cdots\}$ is a probability distribution over ν's with finite mean $L=\sum_{\nu\geq1}\nu p_\nu$ and $P=L^{-1}$.

It is easy to see that $X_n(\omega)=0$ or 1 almost certainly and that for any probability distribution $\{p_\nu; \nu=1, 2, \cdots\}$ with finite mean L there exists a gap process $\{X_n(\omega)\}$ with the gap distribution $\{p_\nu; \nu=1, 2, \cdots\}$. If we define for any finite set of suffixes $(n+1, n+2, \cdots n+k)$ the additive set function $P^*_{n+1,n+2,\cdots n+k}$ on R^k with its value defined for any $A\subset R^k$ by $P^*_{n+1,n+2,\cdots n+k}(A)=\sum_{\substack{\alpha_j\in\{0,1\}\\(\alpha_1\alpha_2\cdots\alpha_k)\in A}} P^*_{n+1,n+2,\cdots n+k}((\alpha_1, \alpha_2, \cdots \alpha_k))$ where $P^*_{n+1,n+2,\cdots n+k}((\alpha_1, \alpha_2, \cdots \alpha_k))$'s are equal to the values given in the above definition of the gap process as $\mathrm{Prob}\{(X_{t+1}(\omega), X_{t+2}(\omega), \cdots X_{t+k}(\omega))=(\alpha_1, \alpha_2, \cdots\alpha_k)\}$, then we get $P^*_{n+1,n+2,\cdots n+k}(R^k)=1$. Taking into account of the fact that $L=\sum_{\mu\geq1}\sum_{\nu\geq\mu}p_\nu$ the proof of this last equation is essentially contained in the following equations

$$\begin{aligned}
&P^*_{n+1,n+2,\cdots n+k+1}((0,0,\cdots 0,0))+P^*_{n+1,n+2,\cdots n+k+1}((0,0,\cdots 0,1))\\
&=P\sum_{\mu>k+1}\sum_{\nu\geqq\mu}p_\nu+P\sum_{\nu\geqq k+1}p_\nu\\
&=P\sum_{\mu>k}\sum_{\nu\geqq\mu}p_\nu\\
&=P^*_{n+1,n+2,\cdots n+k}((0,0,\cdots 0)),\\
&P^*_{n+1,n+2,\cdots n+k+1}((\alpha_1,\alpha_2,\cdots\alpha_k,0))+P^*_{n+1,n+2,\cdots n+k+1}((\alpha_1,\alpha_2,\cdots\alpha_k,1))\\
&=P\Big(\sum_{\nu\geqq\varepsilon_1}p_\nu\Big)p_{\varepsilon_2-\varepsilon_1}p_{\varepsilon_3-\varepsilon_2}\cdots p_{\varepsilon_r-\varepsilon_{r-1}}\Big(\sum_{\mu>k+1-\varepsilon_r}p_\mu\Big)\\
&\qquad\qquad+P\Big(\sum_{\nu\geqq\varepsilon_1}p_\nu\Big)p_{\varepsilon_2-\varepsilon_1}p_{\varepsilon_3-\varepsilon_2}\cdots p_{\varepsilon_r-\varepsilon_{r-1}}p_{k+1-\varepsilon_r}\\
&=P\Big(\sum_{\nu\geqq\varepsilon_1}p_\nu\Big)p_{\varepsilon_2-\varepsilon_1}p_{\varepsilon_3-\varepsilon_2}\cdots p_{\varepsilon_r-\varepsilon_{r-1}}\Big(\sum_{\mu>k-\varepsilon_r}p_\mu\Big)\\
&=P^*_{n+1,n+2,\cdots n+k}((\alpha_1,\alpha_2,\cdots\alpha_k))
\end{aligned}$$

and the similar equations of the type $P^*_{n,n+1,\cdots n+k}((1,\alpha_1,\cdots\alpha_k))+P^*_{n,n+1,\cdots n+k}((0,\alpha_1,\cdots\alpha_k))=P^*_{n+1,n+2,\cdots n+k}((\alpha_1,\alpha_2,\cdots\alpha_k))$. If we define the system of distribution functions $\{F_{\nu_1,\nu_2,\cdots\nu_n}(x_{\nu_1},x_{\nu_2},\cdots x_{\nu_n})\}$ for the set of suffixes of any number n of mutually different integers ν_i's by

$$\begin{aligned}
&F_{\nu_1,\nu_2,\cdots\nu_n}(x_{\nu_1},x_{\nu_2},\cdots x_{\nu_n})\\
&=F_{\mu_1,\mu_2,\cdots\mu_n}(x_{\mu_1},x_{\mu_2},\cdots x_{\mu_n});\ (\mu_1,\mu_2,\cdots\mu_n)\text{ is the rearrangement of }\\
&\qquad\qquad(\nu_1,\nu_2,\cdots\nu_n)\text{ into ascending order of }\nu_i\text{'s}\\
&\equiv P^*_{\mu_1,\mu_1+1,\mu_1+2,\cdots\mu_n}\{(z_{\mu_1},z_{\mu_1+1},z_{\mu_1+2},\cdots z_{\mu_n});\ z_{\mu_i}\leqq x_{\mu_i}\text{ for }i=1,2,\cdots n\}
\end{aligned}$$

we can see from the above stated equations that the system $\{F_{\nu_1,\nu_2,\cdots\nu_n}(x_{\nu_1},x_{\nu_2},\dots x_{\nu_n})\}$ satisfies the consistency relations to assure the existence of a stochastic process with these $F_{\nu_1,\nu_2,\cdots\nu_n}(x_{\nu_1},x_{\nu_2},\cdots x_{\nu_n})$'s as its finite dimensional distribution functions. The process defined by this system is the gap process with the gap distribution $\{p_\nu;\ \nu=1,2,\cdots\}$.

We shall state here a theorem for the use in the later sections.

THEOREM: *For the gap process $\{X_n(\omega)\}$ the generating function $f_k(z)$ of the k-th order $(k\geqq 1)$ factorial moments of $\nu_t(h)\equiv X_{t+1}+X_{t+2}+\cdots X_{t+h}$ is given by the following equation*

$$\begin{aligned}
f_k(z)&=\sum_{h=k}^{\infty}E[\nu_t(h)(\nu_t(h)-1)(\nu_t(h)-2)\cdots(\nu_t(h)-k+1)]z^h\\
&=k!\ P\frac{z}{(1-z)^2}\Big(\frac{p(z)}{1-p(z)}\Big)^{k-1}
\end{aligned}$$

valid for $-1<z<1$, *where* $p(z)=\sum_{\nu\geqq 1} z^{\nu}p_{\nu}$.

PROOF. For $P_h(\lambda)\equiv \text{Prob}\{\nu_t(h)=\lambda\}$ we have

$$\begin{aligned}\sum_{\lambda\geq\nu} P_h(\lambda) &= \text{Prob}\{X_{t+1}(\omega)+X_{t+2}(\omega)+\cdots+X_{t+h}(\omega)\geqq\nu\}\\ &=\sum_{\mu=1}^{h} P\Big(\sum_{\rho\geqq\mu} p_\rho\Big)F^{*\nu-1}(h-\mu) && \nu>1\,,\\ &=\sum_{\mu=1}^{h} P\Big(\sum_{\rho\geqq\mu} p_\rho\Big) && \nu=1\,,\\ &=1 && \nu=0\end{aligned}$$

where $F^{*\nu}(\lambda)=\sum_{\mu=1}^{\lambda} p^{*\nu}(\mu)$ and $p^{*\nu}(\cdot)$ denotes the ν-th convolution of the gap distribution. These equations are easily verified by decomposing the event $(X_{t+1}(\omega)+X_{t+2}(\omega)+\cdots+X_{t+h}(\omega)=\nu)$ into the sum of h events in the μ-th of which the first 1 occurs at the μ-th component from the left and then by applying the probability relations for finite dimensional events stated in the definition of the process.

As we have for $|z|<1$

$$\sum_{\mu\geqq 1} P\Big(\sum_{\rho\geqq\mu} p_\rho\Big)z^{\mu}=\sum_{\mu\geqq 1} P\Big(1-\sum_{\rho<\mu} p_\rho\Big)z^{\mu}=P\frac{z}{1-z}(1-p(z)) \text{ and}$$

$$\sum_{\mu\geqq 1} F^{*\nu}(\mu)z^{\mu}=\sum_{\mu\geqq 1}\Big(\sum_{1\leqq\rho\leqq\mu} p^{*\nu}(\rho)\Big)z^{\mu}=\frac{1}{1-z}p^{\nu}(z)$$

we get

$$\begin{aligned}g_\nu(z)&\equiv\sum_{h=1}^{\infty}\text{Prob}\{X_{t+1}(\omega)+X_{t+2}(\omega)+\cdots+X_{t+h}(\omega)=\nu\}z^{h}\\ &=\frac{z}{1-z}-P\frac{z}{(1-z)^2}(1-p(z)) \quad \nu=0,\\ &=P\frac{z}{(1-z)^2}(1-p(z))^2p^{\nu-1}(z) \quad \nu\geqq 1\,.\end{aligned}$$

Thus we get by using a real u the bivariate generating function

$$g(z,u)=\sum_{\nu=0}^{\infty} g_\nu(z)u^{\nu}=\frac{z}{1-z}-P\frac{z}{(1-z)^2}(1-p(z))+P\frac{z}{(1-z)^2}(1-p(z))^2\frac{u}{1-up(z)}$$

valid for $|z|<1$, $|uz|<1$. We shall here represent by $g_h(u)$ the probability generating function of $\{P_h(\nu)\}$. As $P_h(\nu)=0$ for ν greater than h we have $|g_h(u)|\leqq 1\vee|u|^h$, thus for z and u satisfying $|z|<1$ and $|uz|<1$ the infinite sum $\sum_{h=1}^{\infty} g_h(u)z^h$ converges absolutely and is equal to $g(z,u)$. Now as $\nu_t(h)\leqq h$ holds for any h we have $\left.\frac{\partial^k g_h(u)}{\partial u^k}\right|_{u=1}=E[\nu_t(h)(\nu_t(h)-1)\cdots$

$(\nu_t(h)-k+1)]\leqq h^k$ $(k\geqq 1)$ and $\left|\frac{\partial^k g_h(u)}{\partial u^k}\right|\leqq h^k(1\vee|u|^h)$. Thus for z and u satisfying $|z|<1$ and $|uz|<1$ the infinite sum $\sum_{h=1}^{\infty}\frac{\partial^k g_h(u)}{\partial u^k}z^h$ converges uniformly and absolutely and we have $\frac{\partial^k}{\partial u^k}\Big(\sum_{h=1}^{\infty}g_h(u)z^h\Big)=\sum_{h=1}^{\infty}\Big(\frac{\partial^k}{\partial u^k}g_h(u)\Big)z^h$. Thus we can get for z with $|z|<1$ our desired factorial moment generating function $f_k(z)$ by differentiating the bivariate generating function $g(z, u)$ k-times with respect to u and then putting u equal to 1.

In the former paper [1] we have given a definition of the gap process slightly different from the present one. The present definition does not exclude the case where the primary p_ν's take positive values at only those ν's which are multiples of some fixed positive integer $\tau>1$, while it was excluded in the former definition. When such τ does not exist we call the gap process to be aperiodic and it represents the steady flow of renewals of an article with the life length distribution $\{p_\nu\}$ [1, 8].

We shall hereafter call a random variable following $\{p_\nu\}$ the primary gap. In the gap process we can see from the relation $\sum_{\mu=1}^{\infty}\{P\sum_{\nu\geq\mu}p_\nu p_{\nu_1}p_{\nu_2}\cdots p_{\nu_n}\}=p_{\nu_1}p_{\nu_2}\cdots p_{\nu_n}$ $(n=1, 2, \cdots)$ that the successive gaps of time between 1's after a fixed time make a sequence of mutually independent primary gaps.

REMARK. In § 2 of the paper [1] a proof was given of a theorem which states that the gap process with the gap distribution $\{p_\nu\}$ has continuous spectral density function if the number of ν's with $p_\nu>0$ is finite and the greatest common divisor of those ν's is equal to 1. The proof was given by using the difference equation method but it was incomplete as the case of multiple roots was overlooked in the description. Taking into consideration the case of multiple roots we get the complete description of the proof by using the relation $\sum_{n=0}^{\infty}n^k|z|^n<+\infty$ valid for $|z|<1$. The result of the theorem suggests that for the treatment of the gap process the analysis starting from observations of primary gaps will be pertinent. In this paper we follow this line of approach.

We shall here note that our gap process is reversible in the sense that

$$\begin{aligned}&P((x_{t+1}(\omega), x_{t+2}(\omega), \cdots x_{t+h}(\omega))=(\alpha_1, \alpha_2, \cdots \alpha_h))\\&=P((x_{t-h+1}(\omega), \cdots x_{t-1}(\omega), x_t(\omega))=(\alpha_h \cdots \alpha_2, \alpha_1))\end{aligned}$$

holds for α_i's equal to 1 and 0.

2. Estimation of the variance-time curve

We shall here call the curve of which graph is given by $\{(V(h), h), h=1, 2, \cdots\}$ the variance-time curve of the gap process where $V(h)$ is the variance of $\nu_t(h)$ and is independent of t. Variance-time curve plays an important role in the statistical treatment of the gap process and its use will be discussed in § 3 of this paper. From the result of the theorem in the former section we can get the generating function $V(z)$ of the sequence $\{V(h);\ h=1, 2, \cdots\}$. We have

$$
\begin{aligned}
V(z) &= \sum_{h=1}^{\infty} V(h) z^h \\
&= f_2(z) + f_1(z) - \sum_{h=1}^{\infty} h^2 P^2 z^h \\
&= P \frac{z}{(1-z)^2} \frac{1+p(z)}{1-p(z)} - P^2 \frac{z(1+z)}{(1-z)^3} \\
&= 2P \frac{z}{(1-z)^2} \frac{1}{1-p(z)} - P \frac{z}{(1-z)^2} - P^2 \frac{z(1+z)}{(1-z)^3} \quad \text{for } |z|<1, \\
V(h) &= P(2R(h)-h) - h^2 P^2
\end{aligned}
$$

where $R(h)=R(h-1)+\sum_{\nu=0}^{h-1} P_\nu$ with $R(1)\equiv 1$ and $P_\nu = \sum_{\mu=1}^{\nu} p_\mu P_{\nu-\mu}$ with $P_0 \equiv 1$.

We shall discuss in this section the sampling fluctuation of an estimate $\hat{V}(h)$ of $V(h)$ where $\hat{V}(h)$ is defined by the defining formula of $V(h)$ with $\{p_\nu\}$ replaced by $\{\hat{p}_\nu\}$ the empirical gap distribution.

The main result is given by the following

THEOREM: *If we use as $\{\hat{p}_\nu\}$ the emprical gap distribution obtained from N independent observations of the primary gaps, then under the condition $\sum \nu^2 p_\nu < +\infty$ the distribution function of the randon variable $\sqrt{N}(\hat{V}(h) - V(h))$ tends to the Gaussian distribution function with the zero-mean and the variance D_h^2 as N tends to infinity where D_h^2 is given by the following*

$$
D_h^2 = P^2 \left[\gamma^2(h) \sum_{\nu \geq 1} \nu^2 p_\nu + 4 \left(\sum_{\nu=1}^{h-1} S^2(h-\nu) p_\nu + \gamma(h) \sum_{\nu=1}^{h-1} S(h-\nu) \nu p_\nu \right) \right] - \left[\gamma(h) + 2P \sum_{\nu=1}^{h-1} S(h-\nu) p_\nu \right]^2
$$

where $\gamma(h) = P^2 h^2 - V(h)$ *and* $S(\nu) = \sum_{\mu=0}^{\nu-1} (\nu-\mu) \left(\sum_{\lambda=0}^{\mu} P_{\mu-\lambda} P_\lambda \right)$.

NOTE: The expression for D_h^2 given above is rather complicated and for practical computations the following recurrence formula for $S(\nu)$ will be useful:

$$S(\nu)=S(\nu-1)+\sum_{\mu=0}^{\nu-1}Q_\mu\,,\quad S(0)=0$$

where
$$\begin{aligned} Q_\mu &= \sum_{\lambda=0}^{\mu}P_{\mu-\lambda}P_\lambda \\ &= 2\sum_{\lambda=0}^{(\mu-1)/2}P_{\mu-\lambda}P_\lambda \qquad \text{when } \mu \text{ is an odd number,} \\ &= 2\sum_{\lambda=0}^{(\mu/2)-1}P_{\mu-\lambda}P_\lambda+P_{\mu/2}^2 \quad \text{when } \mu \text{ is an even number.} \end{aligned}$$

Proof. Here we shall use the idea of the differentiable statistical function introduced by Mises [7]. For the sake of simplicity we shall represent by ε_ν the difference $\hat{p}_\nu-p_\nu$ and by $\varepsilon(z)$ the generating function of $\{\varepsilon_\nu\}$ or the difference $\hat{p}(z)-p(z)$, and further by δ the difference between the means of $\{\hat{p}_\nu\}$ and $\{p_\nu\}$ that is $\delta=\sum_{\nu\geq 1}\nu(\hat{p}_\nu-p_\nu)=\sum_{\nu\geq 1}\nu\varepsilon_\nu$. By supplying subscript p to $V(z)$ we shall represent the generating function of $\{V(h)\}$ corresponding to $\{p_\nu\}$. Thus the notations $V_p(z)$, $V_{\hat{p}}(z)$ and $V_{p+t\varepsilon}(z)$ represent the function $V(z)$ with $p(z)$ replaced by $p(z)$, $\hat{p}(z)$, and $p(z)+t\varepsilon(z)$ respectively. The notations $V_p(h)$ $V_{\hat{p}}(h)$ and $V_{p+t\varepsilon}(h)$ should be interpreted accordingly. Taking into account of the relations $|p(z)+t\varepsilon(z)|\leqq|z|$ $(0\leqq t\leqq 1, |z|\leqq 1)$, $|\varepsilon(z)|\leqq|z|$ $(|z|\leqq 1)$ and $L\geqq 1$ we get

$$V_{p+t\varepsilon}(z)=\frac{1}{(L+t\delta)}\frac{1+p(z)+t\varepsilon(z)}{1-p(z)-t\varepsilon(z)}\frac{z}{(1-z)^2}-\frac{1}{(L+t\delta)^2}\frac{z(1+z)}{(1-z)^3}$$

$$\frac{\partial V_{p+t\varepsilon}(z)}{\partial t}=\frac{-\delta}{(L+t\delta)^2}\frac{1+p(z)+t\varepsilon(z)}{1-p(z)-t\varepsilon(z)}\frac{z}{(1-z)^2}+\frac{1}{(L+t\delta)}\frac{2\varepsilon(z)}{(1-p(z)-t\varepsilon(z))^2}\frac{z}{(1-z)^2}+\frac{2\delta}{(L+t\delta)^3}\frac{z(1+z)}{(1-z)^3}$$

$$\frac{\partial^2 V_{p+t\varepsilon}(z)}{\partial t^2}=\frac{2\delta^2}{(L+t\delta)^3}\frac{1+p(z)+t\varepsilon(z)}{1-p(z)-t\varepsilon(z)}\frac{z}{(1-z)^2}-\frac{4\delta}{(L+t\delta)^2}\frac{\varepsilon(z)}{(1-p(z)-t\varepsilon(z))^2}\frac{z}{(1-z)^2}+\frac{4}{(L+t\delta)}\frac{\varepsilon^2(z)}{(1-p(z)-t\varepsilon(z))^3}\frac{z}{(1-z)^2}-\frac{6\delta^2}{(L+t\delta)^4}\frac{z(1+z)}{(1-z)^3}$$

for z and t satisfying the conditions $|z|<\frac{1}{1+a}$ and $-\left(a\wedge\frac{1}{|\delta|}\right)<t<1+\left(a\wedge\frac{1}{|\delta|}\right)$ for some $a>0$. Further as $\frac{\partial^l}{\partial t^l}\left(\frac{\partial^k}{\partial z^k}V_{p+t\varepsilon}(z)\right)=\frac{\partial^k}{\partial z^k}\left(\frac{\partial^l}{\partial t^l}V_{p+t\varepsilon}(z)\right)$ holds for any l and k for t and z satisfying the above stated restrictions we get from the Taylor expansion

$$V_{p+\varepsilon}(z)=V_p(z)+\left(\frac{\partial}{\partial\tau}V_{p+\tau\varepsilon}(z)\right)_{\tau=0}+\frac{1}{2}\left(\frac{\partial^2}{\partial\tau^2}V_{p+\tau\varepsilon}(z)\right)_{\tau=\theta}\quad 0<\theta<1$$

and the expression

$$\left.\frac{\partial V_{p+t\varepsilon}(z)}{\partial t}\right|_{t=0}=P\left\{2P^2\frac{z(1+z)}{(1-z)^3}-P\frac{1+p(z)}{1-p(z)}\frac{z}{(1-z)^2}\right\}\delta+2P\frac{1}{(1-p(z))^2}\frac{z}{(1-z)^2}\varepsilon(z)$$
$$=P\left\{P^2\frac{z(1+z)}{(1-z)^3}-V_p(z)\right\}\delta+2P\frac{1}{(1-p(z))^2}\frac{z}{(1-z)^2}\varepsilon(z)$$

the desired representation

$$\hat{V}(h)=V_{\hat{p}}(h)=V_{p+\varepsilon}(h)=V_p(h)+\Delta_{p,\varepsilon}V(h)+B(h,\,p,\,\varepsilon)$$

where

$$\Delta_{p,\varepsilon}V(h)=P\{h^2P^2-V_p(h)\}\delta+2P\sum_{\nu=1}^{h-1}S(h-\nu)\varepsilon_\nu\ ,\ \text{and}$$

$$B(h,\,p,\,\varepsilon)=\text{coefficients of } z^h \text{ in the expansion of } \left.\frac{\partial^2 V_{p+\tau\varepsilon}(z)}{\partial\tau^2}\right|_{\tau=\theta}.$$

We shall first concern ourselves with the term $\Delta_{p,\varepsilon}V(h)$. As $\delta=\sum_{\nu\geq 1}\nu\varepsilon_\nu$ we have

$$\Delta_{p,\varepsilon}V(h)=P\left[\gamma(h)\sum_{\mu\geq h}\mu\varepsilon_\mu+\sum_{\nu=1}^{h-1}\{2S(h-\nu)+\gamma(h)\nu\}\varepsilon_\nu\right]$$

and thus from the fact that $\hat{p}_\nu$ is given as the ratio of the number of primary gaps of length ν to the total number N of the independent observations we may consider it to be a sample mean $\sum_{i=1}^{N}\Delta V^{(i)}(h)/N$ of N independent random variables $\Delta V^{(i)}(h)$ where $\Delta V^{(i)}(h)$ is defined by the formula for the definition of $\Delta_{p,\varepsilon}V(h)$ with ε_ν's replaced by $\varepsilon_\nu^{(i)}$'s where $\varepsilon_\nu^{(i)}$ takes the values $1-p_\nu$ and $-p_\nu$ according as the i-th primary gap takes the value ν or not.

For such $\varepsilon_\nu^{(i)}$'s the random variable $X=\sum_{\nu\geq 1}a_\nu\varepsilon_\nu^{(i)}$ defined by some $\{a_\nu\}$ satisfying the condition $\sum a_\nu^2 p_\nu<+\infty$ has the mean value zero and the variance given by

$$D^2(X)=\sum_{\nu\geq 1}a_\nu^2p_\nu-(\sum_{\nu\geq 1}a_\nu p_\nu)^2=\sum_{\nu\geq 1}a_\nu^2p_\nu(1-p_\nu)-2\sum_{\nu\geq 1}\sum_{\mu>\nu}a_\nu a_\mu p_\nu p_\mu\ .$$

Thus for $\Delta V^{(i)}(h)$ under the condition $\sum_{\nu\geq 1}\nu^2p_\nu<+\infty$ we have

$$E(\Delta V^{(i)}(h))=0\ ,$$

$$D^2(\Delta V^{(i)}(h))=P^2\left[\gamma^2(h)\sum_{\mu\geq h}\mu^2p_\mu+\sum_{\nu=1}^{h-1}[2S(h-\nu)+\gamma(h)\nu]^2p_\nu\right.$$

$$-\left\{\gamma(h)\sum_{\mu\geqq h}\mu p_\mu+\sum_{\nu=1}^{h-1}[2S(h-\nu)+\gamma(h)\nu]p_\nu\right\}^2\Bigg]$$

$$=P^2\left[\gamma^2(h)\sum_{\nu\geqq 1}\nu^2 p_\nu+4\sum_{\nu=1}^{h-1}(S^2(h-\nu)+\gamma(h)S(h-\nu)\nu)p_\nu\right]$$

$$-\left[\gamma(h)+2P\sum_{\nu=1}^{h-1}S(h-\nu)p_\nu\right]^2$$

and for $\Delta_{p,\varepsilon}V(h)$ we have $E(\Delta_{p,\varepsilon}V(h))=0,\ D^2(\Delta_{p,\varepsilon}V(h))=D^2(\Delta V^{(4)}(h))/N$.

Now as the random variable $\Delta_{p,\varepsilon}V(h)$ is given as an arithmetic mean of the independent random variables $\Delta V^{(4)}(h)$ we can at once see that the distribution function of $\sqrt{N}\Delta_{p,\varepsilon}V(h)$ tends to a Gaussian distribution function with the zero-mean and the variance $D^2(\Delta V^{(4)}(h))$ as N tends to infinity. Thus our proof of the theorem is complete if we can show that the term $B(h, p, \varepsilon)$ tends to be statistically negligible. To show this we shall here give an estimate of the magnitude of $B(h, p, \varepsilon)$. The generating function for $B(h, p, \varepsilon)$ is represented as

$$\frac{z}{(1-z)^2}\left[\left(\frac{2}{(L')^3}\frac{1+p'(z)}{1-p'(z)}-\frac{6}{(L')^4}\frac{1+z}{1-z}\right)\delta^2-\frac{4}{(L')^2}\frac{\varepsilon(z)}{(1-p'(z))^2}\delta+\frac{4}{L'}\frac{\varepsilon^2(z)}{(1-p'(z))^3}\right]$$

by some gap distribution $\{p'_\nu\}$ and its mean L'.

Thus taking into account of the fact that $P_\nu\leqq 1$ and $L\geqq 1$ hold for all ν and for any gap distribution, we have

$$|B(h, p, \varepsilon)|\leqq 8h(h+1)\delta^2+\frac{2}{3}\sum_{\nu=1}^{h-1}(h-\nu)(h-\nu+1)(h-\nu+2)|\varepsilon_\nu||\delta|$$

$$+\frac{1}{6}\sum_{\nu=1}^{h-1}(h-\nu)(h-\nu+1)(h-\nu+2)(h-\nu+3)\sum_{\mu=1}^{\nu-1}|\varepsilon_{\nu-\mu}\varepsilon_\mu|\,.$$

From the inequality $E|XY|\leqq D(X)D(Y)$, valid for any pair of random variables X, Y with the means both equal to zero and variances $D^2(X), D^2(Y)$ respectively, and by the equations

$$D^2(\delta)=\frac{1}{N}\left(\sum_{\nu\geqq 1}\nu^2 p_\nu-L^2\right)\equiv\frac{1}{N}D_g^2\,,\quad D^2(\varepsilon_\nu)=\frac{1}{N}p_\nu(1-p_\nu)\equiv\frac{1}{N}D_\nu^2$$

we have

$$E|B(h, p, \varepsilon)|\leqq\frac{1}{N}\left\{8h(h+1)D_g^2+\frac{2}{3}\sum_{\nu=1}^{h-1}(h-\nu)(h-\nu+1)(h-\nu+2)D_\nu D_g\right.$$

$$\left.+\frac{1}{6}\sum_{\nu=1}^{h-1}(h-\nu)(h-\nu+1)(h-\nu+2)(h-\nu+3)\sum_{\mu=1}^{\nu-1}D_{\nu-\mu}D_\mu\right\}\,.$$

Thus we get

$$E|\sqrt{N}(\hat{V}(h)-V_p(h)-\Delta_{p,\varepsilon}V(h))|\leqq\frac{1}{\sqrt{N}}C(h)$$

for some positive constant $C(h)$.

As the convergence in the mean implies the convergence in law and the convergence in law is equivalent to convergence in some metric space of distribution functions we can see from this last inequality that the distribution function of $\sqrt{N}(\hat{V}(h)-V_p(h))$ converges to the Gaussian distribution function with the zero-mean and the variance $D^2(V^{(1)}(h))$. This completes the proof of the theorem.

Proof of the above stated theorem may be given directly by applying the general theorem of the differentiable statistical function given by Mises [7]. But we have derived our result under slightly weaker condition of existence of the second order moment of the gap distribution, and our approach is simpler for this special case.

We shall here discuss further some properties of variance-time curve at large h. We assume that the process is aperiodic and for a complex z $p(z)=\sum p_\nu z^\nu$ has radius of convergence greater than unity to assure the validity of the following analysis. We define functions $q_i(z)$ for $|z|<1$ by the recurrence relation

$$(1-z)q_i(z)=q_{i-1}(1)-q_{i-1}(z)$$
$$q_0(z)=p(z)$$

where $i!\times q_i(1)$ is defined as the i-th order factorial moment of the gap distribution. Under our assumption $q_i(z)$ tends to $q_i(1)$ as z approaches to 1 along some proper path inside the unit circle. If we use the representation $q_i(z)=\sum_{\nu=1}^{\infty}q_{i,\nu}z^\nu$ we have $q_{i,\nu}=\sum_{\mu=\nu+1}^{\infty}q_{i-1,\mu}$. Using these functions we get the representations

$$\begin{aligned}p(z)&=q_0(1)-(1-z)q_1(1)+(1-z)^2q_2(z)\\ q_2(z)&=q_2(1)-(1-z)q_3(z)\\ &=q_2(1)-(1-z)q_3(1)+(1-z)^2q_4(z)\,.\end{aligned}$$

By these representations we get

$$\frac{1}{1-p(z)}=\frac{1}{(1-z)q_1(1)}\left(1-\left(\frac{1-z}{q_1(1)}\right)q_2(z)\right)^{-1}$$

$$=\frac{1}{(1-z)q_1(1)}\Big[1+\Big(\frac{1-z}{q_1(1)}\Big)\{q_2(1)-(1-z)q_3(1)+(1-z)^2q_4(z)\}$$
$$+\Big(\frac{1-z}{q_1(1)}\Big)^2\{q_2(1)-(1-z)q_3(z)\}^2$$
$$+\Big(\frac{1-z}{q_1(1)}\Big)^3(q_2(z))^3\Big(1-\Big(\frac{1-z}{q_1(1)}\Big)q_2(z)\Big)^{-1}\Big]$$
$$=\frac{1}{(1-z)q_1(1)}\Big[1+(1-z)\frac{q_2(1)}{q_1(1)}-(1-z)^2\frac{q_3(1)}{q_1(1)}+(1-z)^2\Big(\frac{q_2(1)}{q_1(1)}\Big)^2$$
$$+(1-z)^3\Big\{\frac{q_4(z)}{q_1(1)}-2\frac{q_2(1)}{q_1^2(1)}q_3(z)+\frac{(1-z)}{q_1^2(1)}q_3^2(z)$$
$$+\Big(\frac{q_2(z)}{q_1(1)}\Big)^3\Big(1-\Big(\frac{1-z}{q_1(1)}\Big)q_2(z)\Big)^{-1}\Big\}\Big].$$

Thus using $P=(q_1(1))^{-1}$ we have for $|z|<1$

$$V_p(z)=2P\frac{z}{(1-z)^2}\frac{1}{1-p(z)}-P\frac{z}{(1-z)^2}-P^2\frac{z(1+z)}{(1-z)^3}$$
$$=2P^2\frac{z}{(1-z)^2}\Big[\frac{1}{1-z}+Pq_2(1)+(1-z)(P^2q_2^2(1)-Pq_3(1))\Big]-P\frac{z}{(1-z)^2}-P^2\frac{z(1+z)}{(1-z)^3}$$
$$+2P^2z\Big[Pq_4(z)-2P^2q_2(1)q_3(z)+P^2(1-z)q_3^2(z)+P^2q_2^3(z)\frac{1-z}{1-p(z)}\Big].$$

All the members in the second square bracket represent the generating functions of sequences of numbers dominated by some sequence of positive numbers of which infinite sums are convergent. Thus we get the asymptotic expression of $V_p(h)$

$$V_p(h)\sim 2P^2\frac{h(h+1)}{2}+2P^3hq_2(1)+2P^3(Pq_2^2(1)-q_3(1))-Ph-P^2h^2$$
$$=P^2h^2+P^2h+2P^3\frac{(L_2-L_1)}{2}h-Ph-P^2h^2$$
$$+P^4\frac{(L_2-L_1)^2}{2}-P^3\frac{L_3-3L_2+2L_1}{3}$$
$$=P^2h^2+P^2h+hP^3L_2-P^2h-Ph-P^2h^2$$
$$+P^4\frac{3L_2^2-6L_2L_1+3L_1^2-2L_3L_1+6L_2L_1-4L_1^2}{6}$$
$$=hP^3(L_2-L_1^2)+P^4\frac{3L_2^2-2L_3L_1-L_1^2}{6}\equiv\tilde{V}_p(h)$$

where L_i stands for the i-th moment of the gap distribution and $L_1=P^{-1}$ holds, and the sign $\sim$ means that the difference $V_p(h)-\tilde{V}_p(h)$ tends to zero as h tends to infinity.

Now using this asymptotic expression we want to discuss the sampling fluctuation of our estimate at large h. For this purpose we use again the differentiable statistical function method. We get

$$\left.\frac{\partial \tilde{V}_{p+t\varepsilon}(h)}{\partial t}\right|_{t=0}=\frac{\partial}{\partial t}\left(h\frac{L_2+t\delta_2}{(L_1+t\delta_1)^3}-h\frac{1}{(L_1+t\delta_1)}+\frac{1}{2}\frac{(L_2+t\delta_2)^2}{(L_1+t\delta_1)^4}-\frac{1}{3}\frac{(L_3+t\delta_3)}{(L_1+t\delta_1)^3}-\frac{1}{6}\frac{1}{(L_1+t\delta_1)^2}\right)\Bigg|_{t=0}$$

$$=h\frac{\delta_2L_1^3-3\delta_1L_1^2L_2}{L_1^6}+\frac{h}{L_1^2}\delta_1+\frac{2\delta_2L_2L_1^4-4\delta_1L_1^3L_2^2}{2L_1^8}-\frac{1}{3}\frac{\delta_3L_1^3-3\delta_1L_1^2L_3}{L_1^6}+\frac{1}{6}\frac{2\delta_1}{L_1^3}$$

$$=\delta_1\left(-3\frac{L_2}{L_1^4}h+\frac{1}{L_1^2}h-2\frac{L_2^2}{L_1^5}+\frac{L_3}{L_1^4}+\frac{1}{3}\frac{1}{L_1^3}\right)+\delta_2\left(\frac{1}{L_1^3}h+\frac{L_2}{L_1^4}\right)+\delta_3\left(-\frac{1}{3}\frac{1}{L_1^3}\right)$$

where δ_i represents the deviation of the i-th order sample moment from its expected value. Thus using the relations $E(\delta_i\delta_j)=(L_{i+j}-L_iL_j)/N$ (N; size of the sample) we can get approximate value of the variance of the estimate by proceeding entirely analogously as the case of $\hat{V}(h)$. It we put

$$C_1(h)=-3\frac{L_2}{L_1^4}h+\frac{1}{L_1^2}h-2\frac{L_2^2}{L_1^5}+\frac{L_3}{L_1^4}+\frac{1}{3}\frac{1}{L_1^3}$$

$$C_2(h)=\frac{1}{L_1^3}h+\frac{L_2}{L_1^4}$$

$$C_3(h)=-\frac{1}{3}\frac{1}{L_1^3}$$

we can expect that for large h the asymptotic variance of our estimate will be approximated by

$$\frac{1}{N}\tilde{D}_h^2=\frac{1}{N}\{C_1^2(h)(L_2-L_1^2)+C_2^2(h)(L_4-L_2^2)+C_3^2(h)(L_6-L_3^2)+2C_1(h)C_2(h)(L_3-L_2L_1)+2C_1(h)C_3(h)(L_4-L_3L_1)+2C_2(h)C_3(h)(L_5-L_3L_2)\}\ .$$

From the above expression we get

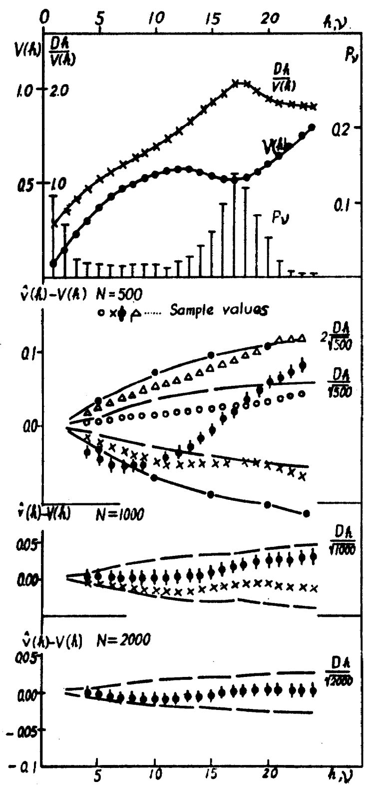
$$\lim_{h\to\infty}\frac{\tilde{D}_h}{V_p(h)}=\frac{\sqrt{\left(1-3\frac{L_2}{L_1^2}\right)^2\frac{L_2-L_1^2}{L_1^2}+2\left(1-3\frac{L_2}{L_1^2}\right)\frac{L_3-L_1L_2}{L_1^3}+\frac{L_4-L_2^2}{L_1^4}}}{\frac{L_2-L_1^2}{L_1^2}}\ .$$

This result will throw some light to see the dependence of the order of fluctuation of our estimate on the characteristic values of the gap distribution. Especially we can see that for large h's the coefficients of variation of our estimates will remain near the same value and the size of the sample of primary gaps will remain nearly the same to guarantee the desired relative accuracy of our estimates. But when we use the sample variance of $\nu_t(h)$ as an estimate of $V(h)$ its coefficient of variation will remain near the value of $\sqrt{2}$ for large h as is expected from the tendency of the distribution of $\nu_t(h)$ to approach to the Gaussian distribution and we have to spend total time length of observation linearly increasing with h. An example of this type of estimate was discussed by Cox and Smith in the paper [5]. Thus for the estimation of $V(h)$ at large h the latter approach will not be practically applicable.

3. Numerical examples

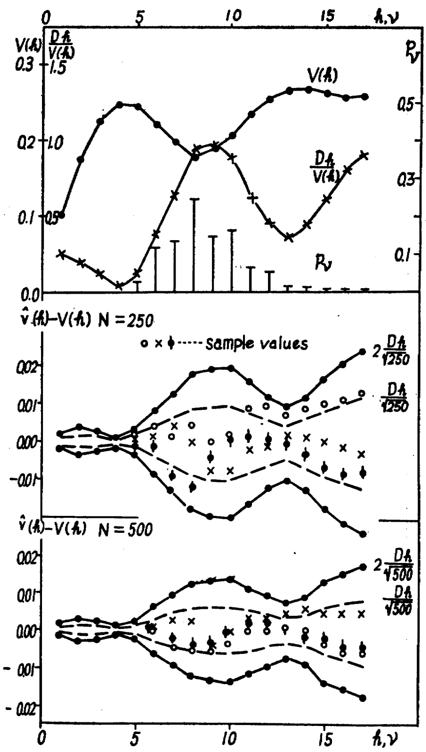
The evaluation of the magnitude of $B(h, p, \varepsilon)$ given in the proof of the theorem in the preceding section is very rough and has little meaning for practical application of the results. Sharper estimate of it will be desirable but it is not easy to get some efficient estimate which will be of practical meaning for arbitrary $\{p_\nu\}$ and finite N. It seems that this is just the place where the Monte Carlo method will be applied with high efficiency. In the former section we have decomposed the random variable $\hat{V}(h)$ into two parts $V_p(h)+\Delta_{p,\varepsilon}V(h)$ and $B(h, p, \varepsilon)$. The first part permits the analytical evaluation of its variance and as N becomes large it is expected that almost the whole fluctuation of our estimate will be due to the fluctuation of $\Delta_{p,\varepsilon}V(h)$. Thus we have only to get some rather rough evaluation of the residual $B(h, p, \varepsilon)$ by Monte Carlo method. This approach will present a typical example of benefitial application of Monte Carlo method and is by far the superior to the heuristic procedure in which we get the estimate of the order of variation of $\hat{V}(h)$ by simply computing the values of $\hat{V}(h)$ for each values of $\{\hat{p}_\nu\}$ using a great number of experimental $\{\hat{p}_\nu\}$'s.

We shall here illustrate part of our numerical results in fig.'s 1∼5. From these examples we can see that at least for these cases the term $B(h, p, \varepsilon)$ seems to be statistically negligible compared with the term $\Delta_{p,\varepsilon}V(h)$ for these N's. Thus our estimate $\hat{V}(h)$ seems to be of practical



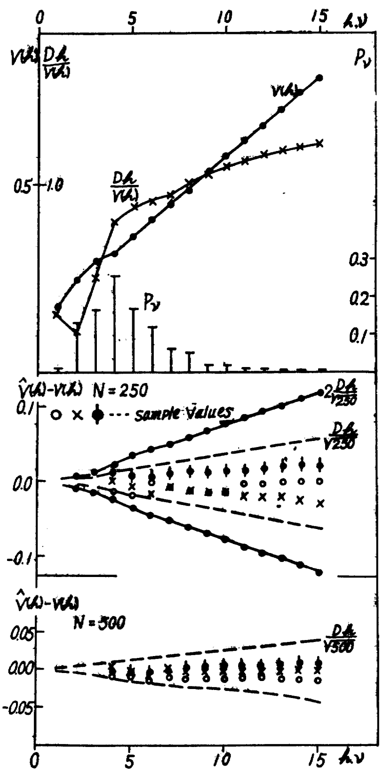
h	$\Delta_{p,\varepsilon}V(h)\times 10^2$	$B(h,p,\varepsilon)\times 10^2$
$N=500$		
8	−4.2515	0.1891
	3.8310	0.1448
	1.3646	0.0186
	−5.6988	0.1302
12	−5.4391	0.2742
	5.3247	0.2989
	1.7572	0.0207
	−3.7198	−0.0813
17	−5.2465	0.3161
	7.7928	0.6428
	2.8235	0.0351
	3.2756	−0.0071
$N=1000$		
8	−0.3278	−0.0021
	−1.8049	0.0289
12	−0.0154	−0.0041
	−2.1025	0.0487
17	1.4360	−0.0054
	−1.8800	0.0495
$N=2000$		
8	−1.0676	0.0100
12	−1.0549	0.0125
17	0.0166	−0.0032

Fig. 1.



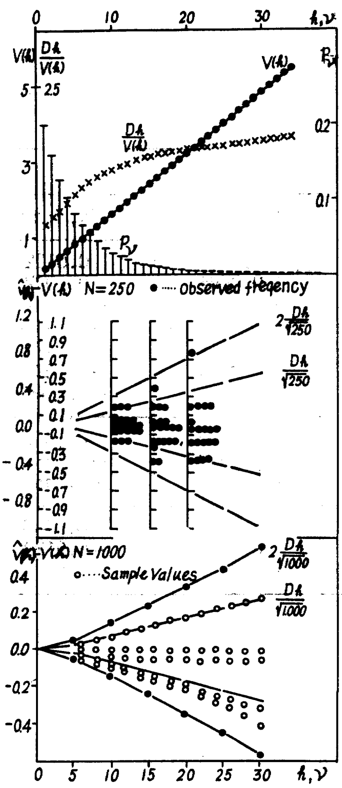
h	$\Delta_{p,\varepsilon}V(h)\times10^3$	$B(h,p,\varepsilon)\times10^3$
$N=250$		
4	−0.0304	0.0002
	−0.2561	−0.0133
	−0.1965	−0.0078
8	−12.3782	0.0114
	0.2558	−0.0131
	4.5776	−0.0340
12	0.7898	0.0231
	−0.9443	−0.0674
	9.8339	−0.0388
16	−9.3311	−0.6265
	−2.0638	0.4723
	11.8368	0.2473
$N=500$		
4	−0.1439	−0.0042
	−0.0481	−0.0056
	−0.2261	−0.0104
8	−6.0530	0.0232
	−5.1625	0.0080
	2.3920	−0.0253
12	−0.0570	0.0108
	2.2041	0.0168
	4.4486	−0.0580
16	−5.0329	−0.0167
	−4.1804	0.1676
	4.8892	0.0510

Fig. 2.



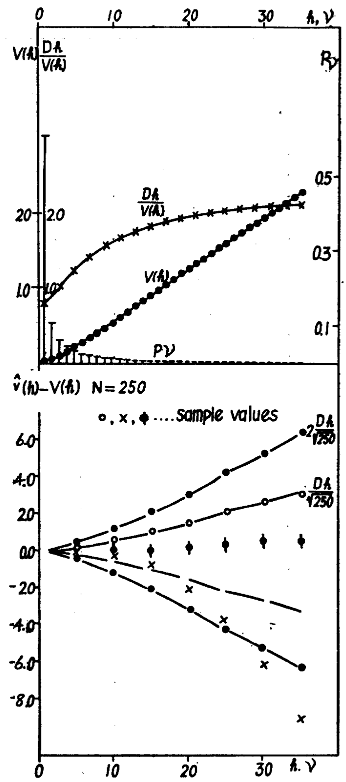
h	$\Delta_{p,\varepsilon}V(h)\times 10^2$	$B(h, p, \varepsilon)\times 10^2$
N=250		
2	−0.1055 0.0126 0.0327	−0.0016 −0.0101 −0.0061
4	−1.8779 0.0150 1.3584	0.0195 −0.0128 −0.0272
6	−1.0025 −1.8933 0.1596	0.0534 0.0150 −0.0196
10	−0.7614 −1.7684 1.5120	−0.0167 0.1432 0.0450
N=500		
2	−0.0471 −0.0319 0.0276	−0.0050 −0.0036 −0.0080
4	0.9198 0.2596 0.6870	0.0053 −0.0033 −0.0215
6	−1.4450 0.4207 −0.8660	0.0436 0.0006 −0.0046
10	1.2556 0.3772 −0.1254	0.0472 0.0001 0.0008

Fig. 3.



N	h	$\Delta_{p,\varepsilon} V(h) \times 10$	$B(h, p, \varepsilon) \times 10$
$N=250$			
	10	−2.1332	0.1327
		−0.8049	0.0330
		−0.3304	−0.0355
		0.2748	0.0008
		−0.8831	−0.0405
		−2.1980	0.1737
		−1.4702	0.0404
		−2.8013	−0.1114
		−0.5859	0.0107
		−0.5925	−0.0062
		−1.4594	−0.0363
		0.9669	0.0757
		0.6120	0.0098
		2.1760	0.3373
		−0.7249	0.0161
		0.0672	−0.0173
		−0.0528	−0.0023
		0.0418	−0.1320
		−0.8697	0.0500
		0.5121	0.0308
$N=1000$			
	10	−0.9403	−0.0101
		−1.0013	−0.0151
		−0.4508	−0.0309
		0.7037	0.0247
		0.0745	0.0132

Fig. 4.



N=250		
h	$\Delta_{p,\varepsilon} V(h)$	$B(h, p, \varepsilon)$
5	0.2127 −0.0594 −0.0896	0.0112 0.0012 −0.0007
10	0.6248 0.0089 −0.2608	0.0475 −0.0021 −0.0046
15	1.0116 0.1012 −0.8609	0.0857 −0.0036 −0.0051
30	2.3631 0.6821 −6.3394	0.2406 −0.0130 0.0404

Fig. 5.

use with moderate size N of the observations of primary gaps in these range of the values of h. As for the estimation of $V(h)$ at large h the results stated at the end of § 2 will be of use and a numerical example is given in fig. 6 for $\{p_\nu\}$ of fig. 8 with $l=2l_0$.

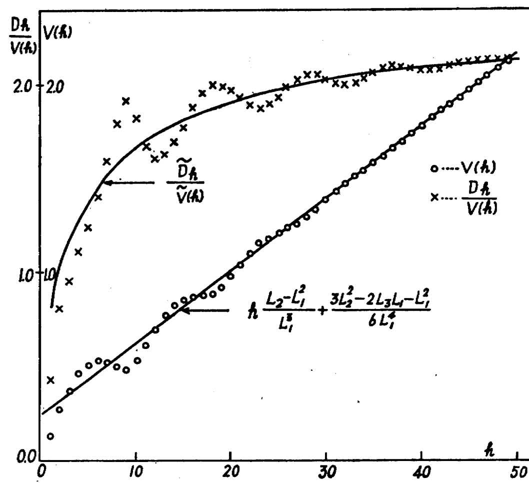


Fig. 6.

We have used random numbers generated by a type of coin-tossings at our institute to obtain the results illustrated in fig.'s 1, 2, 3 and 5. For the example illustrated in fig. 4 a type of quasi-random number was used. The numerical results in this section were obtained by using the FACOM-128 universal relay computer. Generation of the quasi-random numbers and the empirical distributions were performed by the procedure described in the paper [4]. Eight decimal digits representation of number was used throughout the computation and as the agreement of the computational result with the analytically expected result is good as is seen in fig. 6 it seems that it was sufficient enough for the computation of these ranges.

4. Construction of a control system

In this section construction of a statistical control system controlling the gap process is discussed by utilizing the results in the former sections. The model of which we discuss the construction of the control system is taken from the reeling process and the contents of the discussions will be illustrated by some typical numerical example at the end of this section.

We shall concern ourselves with a process which is composed of $K=m\times k$ mutually independent gap processes $\{X_n^{(i)}(\omega)\}$ $i=1, 2, \cdots, K$ following one and the same gap distribution $\{p\}$. We assume that we can make some preliminary observations with the primary gaps to get $\{\hat{p}_\nu\}$ and in the course of the running of the process observations are made on the m processes $\left\{\sum_{\mu=1}^{k} X_n^{(lk+\mu)}(\omega)\right\}$ $(l=0, 1, 2, \cdots, m-1)$.

We shall further assume that our discussion is made under the condition that fairly large k and m are available and $\sum_{\nu\geq 1}\nu^2 p_\nu<+\infty$ holds.

We consider the system where the observations about the process are made simultaneously on the m processes for the time interval of some fixed length h and we get the $\bar{\nu}_t^{(l)}(h)\equiv \sum_{\mu=1}^{k}\nu_t^{(lk+\mu)}(h)$'s$(l=0, 1, \cdots, m-1)$.

To keep the whole process in an ideal state which is determined by the preassigned gap distribution we must devise some statistical control system which will enable us to detect efficiently the change of the gap distribution. For this purpose, under the condition of fairly large k and m, the control system which will detect efficiently the change of $E\{\nu_t(h)\}$ and $D^2\{\nu_t(h)\}$ will be suited.

The problem under consideration is thus reduced to the problem of selecting a proper value of h as the length of the interval of observation. To answer the problem we use the variance-time curve of the gap process.

From the stand point of control of the mean of $\nu_t(h)$ the value of h lying near the h_M which is determined by the following rule 1 will be suitable.

Rule 1. *We define h_M as the smallest integer lying in some preassigned interval $[L,H]$ and satisfying the relation*

$$\operatorname*{Min}_{L\leq h\leq H}\frac{V(h)}{h}=\frac{V(h_M)}{h_M}.$$

The restriction imposed on h_M to be in $[L, H]$ corresponds to the practical limitations for the observations such as those caused by boundary effects and by some other technical conditions.

The rationale for this rule is that when we adopt as our statisitcs the sample mean

$$\frac{1}{m}\sum_{i=0}^{m-1}\bar{\nu}_i^{(e)}(h)$$

with m satisfying the condition $hm \doteqdot$ const to control the mean $E\{\nu_t(h)\}$ it shows minimum coefficient of variation at $h=h_M$ i.e. under the condition that the whole time length elapsed during the observation is constant our statistics shows minimum coefficient of variation by taking the interval length of observation equal to h_M.

Next from the stand point of the control of the variance of $\nu_t(h)$ the value of h lying near the h_V which is determind by the following rule 2 will be a suitable one.

Rule 2. *We define h_V as the smallest integer in the preassigned interval $[L, H]$ satisfying the relation*

$$\underset{L\leq h\leq H}{\text{Max}}\frac{D_h}{V(h)}=\frac{D_{hV}}{V(h_V)}$$

where D_h is given in the theorem of section 2.

The rationale for this rule is that if we suppose that the deviation of the gap distribution from the preassigned one occurs in such a way as the $\{\hat{p}_\nu\}$ with some N deviates from $\{p_\nu\}$ then the ratio $D_h/V(h)$ may be taken as an index which shows the sensitivity at h of the variance-time curve for the deviation of the gap distribution. Thus it may be expected that the deviation of the variance time curve will be most significant at $h=h_V$ for ordinary deviation of the gap distribution. Obviously this reasoning will only be a crude approximation to the realities and precise determination of the pattern of deviations of the gap distribution will allow us to set up more appropriate procedures. We may also adopt the rule 2′ the modified one of rule 2 in which the definition of h_V is given as h'_V which is the smallest value of h maximizing the ratio $D_h/\sqrt{h}\,V(h)$ and satisfying $L\leqq h\leqq H$. This comes from the consideration that, under the condition of large k, our $\bar{\nu}_i^{(1)}(h)$ may be expected to be nearly normally distributed and thus the ratio $[E\{(\bar{\nu}_i^{(1)}(h)-hP)^4\}-(V(h))^2]/(V(h))^2$ will be nearly equal to 2 for every h in the range of consideration and so if we adopt the statistics

$$\frac{1}{m}\sum_{i=0}^{m-1}(\bar{\nu}_i^{(1)}(h)-kP)^2$$

as an estimate of the $V(h)$ it has the coefficient of variation nearly equal to $\sqrt{2/m}$ and thus under the condition $mh \doteqdot c$ (constant) it is nearly equal to $\sqrt{2h/c}$ and the ratio $D_h/\sqrt{h}\,V(h)$ may be considered to show the sensitivity of the curve at h under the restriction of constant total observation time. But as the value $D_h/V(h)$ has not so definite meaning as $V(h)/h$ so the values $D_h/V(h)$ and $D_h/\sqrt{h}\,V(h)$ seem a little unreliable and it will be reasonable to adopt an h lying near h_M as our observation interval length h_0 also taking into account of the values h_V and h'_V.

From the relation $E\{\nu_t(h)\}=hP$ we can see that even in the case where the types of deviations of gap distribution from the preassigned one are precisely known the above stated procedure will be still of use if we adopt as h_M that defined by rule 1 and h_V and h'_V those defined by the rules 2 and 2′ with D_h replaced by some appropriate constant which corresponds to the practically signifiant deviation of the variance from the preassigned value. Thus for example if the typically deviated gap distribution is represented by a fixed geometric distribution

$$\{p(1-p)^{\nu-1};\ \nu=1, 2, \cdots\} \qquad 0<p<1$$

D_h should be relaced by $|hp(1-p)-V(h)|$.

Adoption of rules 2 and 2′ stated above seems to show an interesting interpretation of the use of differentiable statistical function, and this interpretation seems to be especially suited for the treatment of the sequence $\{V(h)\}$. Of course such an interpretation is only necessary when we have not enough knowledge about the typical deviations of the gap distribution and it is very much similar to the analysis of the errors due to roundings in the numerical computation using random variables as an approximate representations of errors.

Fig. 7 shows necessary informations for this procedure for a concrete gap distribution. This gap distribution is an approximation to an nonbreaking length distribution of baves. In this example if we take into consideration the fact that the value $V(h)/h$ is nearly equal for $h=35$ and 36 it seems reasonable to take h_0 in the range [33, 36] the value near the mode of the gap distribution. This result was found to give a complete explanation of the observation length which had been empirically recognized to be the best suited for the purpose of controlling the reeling process.

Of course practical utility of our procedure largely depends on the

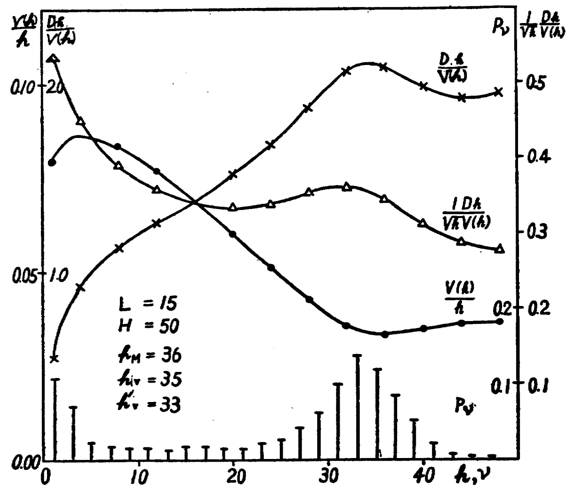


Fig. 7.

type of the gap distribution, but the numerical results illustrated in fig.'s 1∼5 in the former section seem to suggest the wide applicability of our procedure. It seems that our procedure is especially usefull to those gap processes of which gap distributions have mode different from 1.

Further it must be noticed that our main concern here is with the case where H is rather small and the fine structure of $\{p_\nu\}$ is reflected in the shape of variance-time curve in that range of h.

Thus our statistical control system is set up by the following procedure.

0. Determine the values L and H from practical conditions of the process under consideration.
1. First compute a graph $\{(\hat{V}(h), h)\, h=1, 2, 3, \cdots\}$ using some $\{\hat{p}_\nu\}$. $\{\hat{p}_\nu\}$ is determined from a sample of size N which is obtained by the preliminary observations made about the primary gaps. The size N here must be large enough to assure the values $\frac{1}{\sqrt{N}}\frac{D_h}{V(h)}$ to be smaller than some fixed constant for h's in the range of our consideration. Usually we can get the range of the values $D_h/V(h)$ from possible types of $\{p_\nu\}$'s under consideration.
2. Compute the values $\frac{\hat{V}(h)}{h}$, $\frac{\hat{D}_h}{V(h)}$, $\frac{1}{\sqrt{h}}\frac{\hat{D}_h}{V(h)}$ $(L \leqq h \leqq H)$ where $\hat{D}_h$ donotes the value of D_h corresponding to the values $\hat{p}_\nu$ of p_ν.

3. Find the values h_M, h_V, and h'_V by rules 1, 2 and 2′ and then decide the interval length h_0 of observation.
4. Detemine the value of m which satisfies the conditions

$$\alpha\frac{1}{\sqrt{mk}}\sqrt{\hat{V}(h_0)}\leqq d_M(h_0)\,, \qquad \beta\frac{\sqrt{2}\,\hat{V}(h_0)}{\sqrt{m}}\leqq d_V(h_0)\,,$$

where α and β may be taken to be equal to 3 or 4 and $d_M(h_0)$, $d_V(h_0)$ are given as the values showing the order of deviation of mean and variance of $\nu_t(h_0)$ which correspond to the practically or technically significant deviation of the gap distribution. Determination of values of α, and β may be made rationally by taking into account of the precise knowledge about the distributions of our statistics which are defined in the following step.
5. Make an observation of length h_0 about m independent processes where m is determined in 4. Compute the statistics

$$\bar{\nu}_t(h_0)=\frac{1}{km}\sum_{l=0}^{m-1}\bar{\nu}_t^{(l)}(h_0) \qquad \text{and}$$

$$s_t^2(h_0)=\frac{1}{m}\sum_{l=0}^{m-1}\left(\frac{1}{k}\bar{\nu}_t^{(l)}(h_0)\right)^2-(\bar{\nu}_t(h_0))^2.$$

Decide that there is some disorder in the condition of the process if at least one of the inequalities

$$|\bar{\nu}_t(h_0)-h_0\hat{P}|>\frac{\alpha}{2}\frac{1}{\sqrt{mk}}\sqrt{\hat{V}(h_0)}$$

$$|ks_t^2(h_0)-\hat{V}(h_0)|>\frac{\beta}{\sqrt{2}}\;\frac{1}{\sqrt{m}}\;\hat{V}(h_0)$$ holds, where $\hat{P}$ represents the value of P corresponding to $\{\hat{p}_\nu\}$.

In the above description we are taking into consideration that k is fairly large and the distribution of $\bar{\nu}_t^{(l)}$'s are expected to be nearly Gaussian. Of course even without such conditions we can apply our system by properly setting the constants α, β, $d_M(h_0)$ and $d_V(h_0)$. We can get necessary informations to calculate the moments of statistics $s_t^2(h_0)$ from the result of the theorem in section 1.

5. Comments on the practical application

We have treated in this paper the gap process with discrete time parameter but such a process is sometimes used as an approximation to a continuous time parameter process. In such cases care should be taken not to introduce a type of systematic bias as is explained in the following. To define the approximate gap distribution $\{p_\nu\}$ we are tempted as was suggested by Feller [6] to define p_ν's by $p_\nu=\int_{(\nu-1)l}^{\nu l} dF(x) \quad (\nu=1, 2, \cdots)$ using some proper small constant length l and the distribution function $F(x)$ of the original gaps. As is easily seen the computations needed to evaluate the $V(h)$'s and D_h^2's become more time consuming as the value of l becomes smaller. Thus it is desirable to take l as big as possible. Such a consideration is especially necessary to make our procedure practical in the case where the original gap distribution is frequently replaced by some new ones. This is just the case of the reeling process where the gap distribution is replaced for each lot of material cocoons. In these circumstances above definition of p_ν is inadequate and we should define a gap distribution as follows:

$$p_{2\nu-1}=\int_{(\nu-1)l}^{\nu l} dF(x) \quad \nu=1, 2, \cdots$$
$$p_{2\mu}=0 \qquad \mu=1, 2, \cdots .$$

Of course to apply the results obtained from this gap process to the original process we should interpret them in the scale of $l/2$. More appropriate approximations will be available, but the above stated one seems to be of practical use. Numerical examples illustrated in fig.'s 8 and 9 will suffice to show how a big systematic bias may be introduced by the former definition.

Numerical examples illustrated in fig.'s 8 and 9 are obtained by using these two types of approximations of $F(x)$, which is an empirical distribution function obtained by Shimazaki from observations of non-breaking lengthes of baves, with $l=l_0$, $2l_0$ and $3l_0$ where l_0 is a properly chosen constant. These examples will also suggest the way how to select a value of l appropriate for practical applications. In the paper [3] we have observed that for the purpose of approximating the density function it will be most satisfactory from some points of view to group the observed data into 10 to 20 intervals for some typical types of distributions. It seems that such order of the number of groups is satisfactory for this case too.

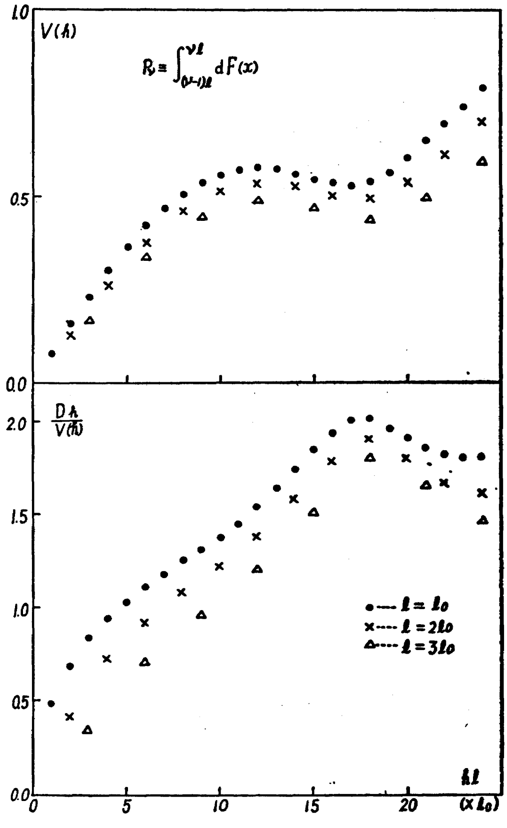


Fig. 8.

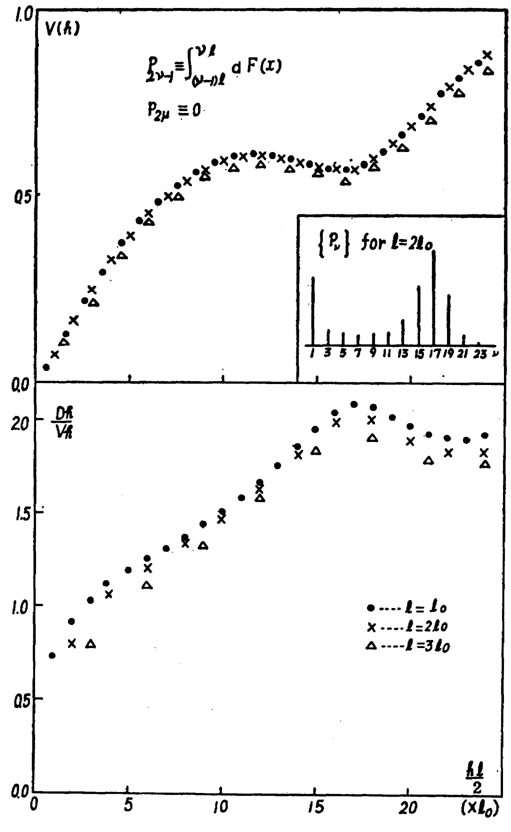


Fig. 9.

Acknowledgement

I wish to express my thanks to Mr. A. Shimazaki, who by supplying typical experimental results of the reeling process stimulated me to the present investigation, and to Mr. M. Motoo for reading the draft of the paper. I also wish to thank Miss Y. Saigusa who has performed the numerical computations by FACOM-128 automatic relay computer.

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ERRATA

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Page	Line	Read	Instead of
228	18	empirical	emprical
233	29	fundamental	foundamental
239	17	$\vert\varepsilon(z)\vert \leqq 2\vert z\vert$	$\vert\varepsilon(z)\vert \leqq \vert z\vert$
〃	23	$2\vert z\vert < \frac{1}{1+a}$	$\vert z\vert < \frac{1}{1+a}$
240	1	h	z
〃	9	coefficient	coefficients
〃	9	$2B(h,\ p,\ \varepsilon)$	$B(h,\ p,\ \varepsilon)$
241	17, 24	$2B(h,\ p,\ \varepsilon)$	$B(h,\ p,\ \varepsilon)$
〃	12	function $\left.\frac{\partial^2 V_{p+\tau\varepsilon}(z)}{\partial\tau^2}\right\vert_{\tau=\theta}$	generating function for $B(h,\ p,\ \varepsilon)$
242	25	$\sum_{\nu=0}^{\infty} q_{\iota,\nu} z^{\nu}$	$\sum_{\nu=1}^{\infty} q_{\iota,\nu} z^{\nu}$
252	3	$\{p_{\iota}\}$	$\{p\}$
253	1, 29, 31	$\frac{1}{k}\bar{\nu}_t^{(e)}(h)$	$\bar{\nu}_t^{(e)}(h)$
〃	29, 32	$\frac{1}{k}V(h)$	$V(h)$
269	29	$W(t)\Delta t$	$W(t)dt$
271	27	$+\gamma\vert(h_1(\tau)-h_2(\tau)\vert\cdot\int_0^t h \cdots$	$+\gamma\vert(h_1(\tau)-h_2(\tau)-\int_0^t h \cdots$
〃	29	$\leqq(K_1+K_2\eta)\sup_{0<u<t}\vert+$	$\leqq(K_1+K_2\eta)\sup_{0<u\wedge t}\vert+$