ORTHOGONAL POLYNOMIALS WITHOUT CONSTANT TERM

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1. On the problem.

Let $X_n^{(N)}(x)$ $(n=1, 2, \cdots, N)$ be polynomials in x of precise degree n, which have not constant terms and are orthogonal with respect to summation at equidistant N points (which we may take $x=1, 2, \cdots, N$ without loss of generality):

$$X_n^{(N)}(0)=0\ , \tag{1}$$

$$\sum_{x=1}^{N} X_n^{(N)}(x)X_m^{(N)}(x)=0\ , \qquad n\neq m. \tag{2}$$

The above conditions define $X_n^{(N)}(x)$ uniquely except for a multiplier.

These polynomials may be of use in fitting curves which pass the origin as Chebyshev-Fisher polynomials (see e.g. [1]) are so in fitting general curves. That is, estimating the coefficients of regression model of the form

$$Y(x)=\beta_1^*x+\beta_2^*x^2+\cdots+\beta_n^*x^n+\varepsilon_x \tag{3}$$

(ε_x are independent and have the same variance) based on observed values $y(1), y(2), \cdots, y(N)$, we transform $Y(x)$ into the form

$$Y(x)=\beta_1X_1^{(N)}(x)+\beta_2X^{(N)}(x)+\cdots+\beta_nX_n^{(N)}(x)+\varepsilon_x\ , \tag{4}$$

then the least-square estimates of β_i are

$$b_i=\frac{B_i}{S_i^{(N)}} \qquad (i=1, 2, \cdots, n)\ , \tag{5}$$

where

$$\begin{aligned} B_i&=\sum_{x=1}^{N} y(x)X_i^{(N)}(x)\ , \\ S_i^{(N)}&=\sum_{x=1}^{N}\{X_i^{(N)}(x)\}^2\ . \end{aligned} \tag{6}$$

Their variances are

$$V(b_i)=\frac{\sigma^2}{S_i^{(N)}}\ , \qquad \sigma^2=V(\varepsilon_x) \tag{7}$$

and σ^2 is estimated by

$$\frac{S_E}{N-n-1}=\frac{1}{N-n-1}\left[\sum_{x=1}^{N} y^2(x)-\sum_{i=1}^{n}\frac{B_i^2}{S_i}\right]. \tag{8}$$

The following Table 1 gives the numerical values of $X_n^{(N)}(x)$, $n=1, 2, 3, 4$, $N=2(1)23$, $x=1(1)N$, with the values of $S_i^{(N)}$. Here we determined the indefinite multipliers of $X_n^{(N)}(x)$ so that the tabulated values be integers as simple as possible. To facilitate to get the regression equation of polynomial form, coefficients of $X_n^{(N)}(x)$ are also tabulated in Table 2.

2. General forms of the polynomials.

Table 1 and 2 are computed from the expressions

$$\begin{aligned}
X_1^{(N)}(x)&=x\ ,\\
X_2^{(N)}(x)&=\lambda_2^{(N)}\left\{x^2-\frac{3N(N+1)}{2(2N+1)}x\right\},\\
X_3^{(N)}(x)&=\lambda_3^{(N)}\left\{x^3-\frac{2N(N+1)(2N+1)}{3N^2+3N+2}x^2\right.\\
&\qquad\left.+\frac{6N^4+12N^3+3N^2-3N+2}{5(3N^2+3N+2)}x\right\},\\
X_4^{(N)}(x)&=\lambda_4^{(N)}\left\{x^4-\frac{15N(N+1)(N^2+N+2)}{4(2N+1)(N^2+N+3)}x^3\right.\\
&\qquad+\frac{5(3N^4+6N^3+5N^2+2N+6)}{14(N^2+N+3)}x^2\\
&\qquad\left.-\frac{5N(N+1)(2N^4+4N^3+N^2-N+18)}{28(2N+1)(N^2+N+3)}x\right\},
\end{aligned} \tag{9}$$

where $\lambda_n^{(N)}$ $(n=2, 3, 4)$ are approporiate multipliers.

$S_n^{(N)}$ may be computed directly from the equation

$$\begin{aligned}
S_1^{(N)}&=\frac{(N+1)^{(2)}(2N+1)}{6}\ ,\\
\frac{S_2^{(N)}}{\{\lambda_2^{(N)}\}^2}&=\frac{(N+2)^{(4)}(3N^2+3N+2)}{120(2N+1)}\ ,\\
\frac{S_3^{(N)}}{\{\lambda_3^{(N)}\}^2}&=\frac{(N+3)^{(6)}(2N+1)(N^2+N+3)}{1050(3N^2+3N+2)}\ ,\\
\frac{S_4^{(N)}}{\{\lambda_4^{(N)}\}^2}&=\frac{(N+4)^{(8)}(5N^4+10N^3+55N^2+50N+24)}{70560(2N+1)(N^2+N+3)}\ .
\end{aligned} \tag{10}$$

The notation $Z^{(\alpha)}$ in right hands are factorial products.

Put

$$\begin{aligned} X_n^{(N)}(x) &= \lambda_n^{(N)} p_n^{(N)}(x) \\ &= \lambda_n^{(N)} \{x^n + c_{n\,n-1} x^{n-1} + \cdots + c_{n1} x\} \; . \end{aligned} \tag{11}$$

The coefficients c's are solutions of the simultaneous equations

$$\begin{aligned} &c_{n1} T_2 + c_{n2} T_3 + \cdots + c_{n\,n-1} T_n + T_{n+1} = 0 \\ &c_{n1} T_3 + c_{n2} T_4 + \cdots + c_{n\,n-1} T_{n+1} + T_{n+2} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &c_{n1} T_n + c_{n2} T_{n+1} + \cdots + c_{n\,n-1} T_{2n-2} + T_{2n-1} = 0 \; , \end{aligned} \tag{12}$$

where

$$T_n = \sum_{x=1}^{N} x^n \; .$$

$p_n^{(N)}(x)$ satisfies, as general orthogonal polynomials, a recurrence formula

$$p_n^{(N)}(x) = \left(x - \frac{\sum_{x=1}^{N} x \{p_{n-1}^{(N)}(x)\}^2}{\sum_{x=1}^{N} \{p_{n-1}^{(N)}(x)\}^2} \right) p_{n-1}^{(N)}(x) - \frac{\sum_{x=1}^{N} \{p_{n-1}^{(N)}(x)\}^2}{\sum_{x=1}^{N} \{p_{n-2}^{(N)}(x)\}^2} p_{n-2}^{(N)}(x) \; . \tag{13}$$

Note that

$$\begin{aligned} &\sum_{x=1}^{N} \{p_n^{(N)}(x)\}^2 = \sum_{x=1}^{N} x^n p_n^{(N)}(x) = S_n^{(N)} / \lambda_n^{(N)} \; , \\ &\sum_{x=1}^{N} x \{p_n^{(N)}(x)\}^2 = \sum_{x=1}^{N} x^{n+1} p_n^{(N)}(x) + c_{n\,n-1} \sum_{x=1}^{N} \{p_n^{(N)}(x)\}^2 \; . \end{aligned} \tag{14}$$

A more general expression of $p_n^{(N)}(x)$ may be obtained according to Chebyshev [2]. In the following we write in brief $p_n(x)$ for $p_n^{(N)}(x)$.

Let $t_i(x)$ be any polynomial of degree $i<n$ without constant term. The conditions (1), (2) are equivalent to

$$p_n(0) = 0 \; , \tag{15}$$

$$\sum_{x=1}^{N} p_n(x) t_i(x) = 0 \qquad i = 1, 2, \cdots, n-1 \; . \tag{16}$$

We assume $p_n(x)$ to be expressed as an n-th difference of a polynomial $P(x)$ of degree $2n$:

$$p_n(x) = \Delta^n P(x) \; . \tag{17}$$

The left hand of (16) is expressed as

$$
\begin{aligned}
&\sum_{x=0}^{N} t_i(x)\cdot\Delta^n P(x)\\
&\quad=-t_i(0)\cdot\Delta^{n-1}P(0)+\Delta t_i(0)\cdot\Delta^{n-2}P(1)-\Delta^2 t_i(0)\cdot\Delta^{n-3}P(2)\\
&\qquad\qquad+\cdots+(-1)^n\cdot\Delta^{n-1}t_i(0)P(n-1)\\
&\qquad+t_i(N)\cdot\Delta^{n-1}P(N+1)-\Delta t_i(N-1)\cdot\Delta^{n-2}P(N+1)\\
&\qquad\qquad+\Delta^2 t_i(N-2)\cdot\Delta^{n-3}P(N+1)\\
&\qquad\qquad-\cdots-(-1)^n\cdot\Delta^{n-1}t_i(N-n+1)P(N+1)\\
&\qquad\qquad+(-1)^n\sum_{x=0}^{N-n}\Delta^n t_i(x)P(x+n)\,, \qquad (18)
\end{aligned}
$$

and $t_i(x)$ satisfied $t_i(0)=\Delta^n t_i(x)=0$. So that, a sufficient condition for (16) is

$$
\begin{aligned}
&\Delta^{n-2}P(1)=\Delta^{n-3}P(2)=\cdots=P(n-1)\\
&=\Delta^{n-1}P(N+1)=\Delta^{n-2}P(N+1)=\cdots=P(N+1)=0\,, \qquad (19)
\end{aligned}
$$

or

$$
\begin{aligned}
&P(1)=P(2)=\cdots=P(n-1)\\
&=P(N+1)=P(N+2)=\cdots=P(N+n)=0\,. \qquad (20)
\end{aligned}
$$

Then, except for a multiplier

$$P(x)=(x+\alpha)(x-1)^{(n-1)}(x-N-1)^{(n)}\,. \qquad (21)$$

The constant α should be determined from the condition (15), i.e.

$$\Delta^n P(0)=0\,,$$

and from this finally

$$
\begin{aligned}
&p_n(x)=\Delta^n P(x)\,,\\
&P(x)=\left\{x+\frac{nN^{(n)}}{(N+n)^{(n)}-N^{(n)}}\right\}(x-1)^{(n-1)}(x-N-1)^{(n)}\,. \qquad (22)
\end{aligned}
$$

3. The case of heterogeneous variances.

In the discussions of section 1 and 2, we postulated the homogeneity of the error. When the standard deviation $D(\varepsilon_x)$ are heterogeneous and, in special, proportional to x as in many practical cases, we need not new polynomials. Indeed, the model

$$Y(x)=\beta_1^* x+\beta_2^* x^2+\cdots+\beta_n^* x^n+x\varepsilon_x$$

is equivalent to

$$\frac{Y(x)}{x}=\beta_1^*+\beta_2^* x+\cdots+\beta_n^* x^{n-1}+\varepsilon_x\,,$$

where $D(\varepsilon_x)$'s are equal. Making use of usual Chebyshev-Fisher polynomials the regression coefficients are computed, with the necessary transformation, by the same abbreviated algorithm.

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REFERENCES

[1] Anderson, R. L. and E. E. Houseman, Tables of orthogonal polynomial values extended to $N=104$, Agriculture Experiment Station, Iowa State College, Research Bulletin 297, 1942.

[2] Chebyshev, P. L., "Sur l'interpolation", Works (Russian edition), Vol. 2, 358–374, 1947, AN USSR.

Table 1. Orthogonal polynomial values

$N=2$

X	X_2
1	−2
2	1
5	5

$N=3$

X_1	X_2	X_3
1	−11	3
2	− 8	−3
3	9	1
14	266	19

$N=4$

X_1	X_2	X_3	X_4
1	−7	71	−4
2	−8	−3	6
3	−3	−67	−4
4	8	34	1
30	186	10695	69

$N=5$

X_1	X_2	X_3	X_4
1	−17	103	−379
2	−23	46	256
3	−18	−56	246
4	− 2	−88	−374
5	25	65	125
55	1771	27830	425194

$N=6$

X_1	X_2	X_3	X_4
1	−25	47	−99
2	−37	35	17
3	−36	−4	82
4	−22	−38	12
5	5	−35	−95
6	45	37	41
91	5824	7488	27664

$N=7$

X_1	X_2	X_3	X_4
1	−23	37	−879
2	−36	35	−149
3	−39	11	587
4	−32	−18	552
5	−15	−35	−205
6	12	−23	−809
7	49	35	441
140	7140	6018	2335102

$N=8$

X_1	X_2	X_3	X_4
1	−91	329	−2177
2	−148	359	−907
3	−171	199	933
4	−160	−42	1656
5	−115	−255	765
6	−36	−331	−1047
7	77	−161	−1897
8	224	364	1288
204	155652	611490	16113790

$N=9$

X_1	X_2	X_3	X_4
1	−58	1358	−3398
2	−97	1631	−2053
3	−117	1159	627
4	−118	282	2412
5	−100	−660	2250
6	−63	−1327	267
7	−7	−1379	−2233
8	68	−476	−2768
9	162	1722	2322
285	85272	13217160	45145672

$N=10$

X_1	X_2	X_3	X_4
1	−24	2118	−966
2	−41	2726	−726
3	−51	2239	−41
4	−54	1072	554
5	−50	−360	750
6	−39	−1642	464
7	−21	−2359	−161
8	4	−2096	−756
9	36	−438	−726
10	75	3030	750
385	19173	40235910	4233658

$N=11$

X_1	X_2	X_3	X_4
1	−175	1263	−417
2	−304	1713	−363
3	−387	1549	−103
4	−424	970	167
5	−415	175	320
6	−360	−637	298
7	−259	−1267	112
8	−112	−1516	−158
	81	−1185	−363
10	320	−75	−285
11	605	2013	363
506	1309022	17671797	917631

$N=12$

X_1	X_2	X_3	X_4
1	−209	363	−15279
2	−368	513	−14799
3	−477	497	−6561
4	−536	362	3289
5	−545	155	10460
6	−504	−77	12516
7	−413	−287	8876
8	−272	−428	814
9	−81	−453	−8541
10	160	−315	−14205
11	451	33	−9339
12	792	638	14751
650	2352350	1781065	1431369960

$N=13$

X_1	X_2	X_3	X_4
1	−41	1265	−50809
2	−73	1848	−53394
3	−96	1886	−30064
4	−110	1516	1534
5	−115	875	28415
6	−111	100	42256
7	−98	−672	39396
8	−76	−1304	20836
9	−45	−1659	−7761
10	−5	−1600	−36070
11	44	−990	−49104
12	102	308	−27214
13	169	2431	53911
819	112203	26095212	18337315136

$N=14$

X_1	X_2	X_3	X_4
1	−143	8593	−405691
2	−257	12901	−454751
3	−342	13714	−298716
4	−398	11822	−60296
5	−425	8015	166625
6	−423	3083	316989
7	−392	−2184	354564
8	−332	−6996	271944
9	−243	−10563	90549
10	−125	−12095	−139375
11	22	−10802	−338756
12	198	−5894	−399696
13	403	3419	−185471
14	637	17927	469469
1015	1667848	1437923280	1349103301512

$N=15$

X_1	X_2	X_3	X_4
1	−329	22841	−208663
2	−596	35087	−246493
3	−801	38543	−180613
4	−944	35014	−66428
5	−1025	26305	52375
6	−1044	14221	143827
7	−1001	567	187677
8	−896	−12852	175392
9	−729	−24231	110157
10	−500	−31765	6875
11	−209	−33649	−107833
12	144	−28078	−195628
13	559	−13247	−206453
14	1036	12649	−78533
15	1575	51415	261625
1240	10653832	12019805680	410065159788

$N=16$

X_1	X_2	X_3	X_4
1	−125	5957	−26793
2	−228	9331	−33059
3	−309	10531	−26299
4	−368	9966	−12804
5	−405	8045	2345
6	−420	5177	15277
7	−413	1771	23331
8	−384	−1764	25056
9	−333	−5019	20211
10	−260	−7585	9765
11	−165	−9053	−4103
12	−48	−9014	−18004
13	91	−7059	−27339
14	252	−2779	−26299
15	435	4235	−7865
16	640	14392	36192
1496	1835592	959096820	7732962630

$N=17$

X_1	X_2	X_3	X_4
1	−212	764	−5588
2	−389	1217	−7153
3	−531	1405	−6069
4	−638	1374	−3504
5	−710	1170	−420
6	−747	839	2427
7	−749	427	4487
8	−716	−20	5416
9	−648	−456	5076
10	−545	−835	3535
11	−407	−1111	1067
12	−234	−1238	−1848
13	−26	−1170	−4524
14	217	−861	−6069
15	495	−265	−5385
16	808	664	−1168
17	1156	1972	8092
1785	6240360	18364488	382992728

$N=18$

X_1	X_2	X_3	X_4
1	-238	4828	-55828
2	-439	7804	-73748
3	-603	9185	-65913
4	-730	9228	-42774
5	-820	8190	-13080
6	-873	6328	16122
7	-889	3899	39487
8	-868	1160	53372
9	-810	-1632	55836
10	-715	-4220	46640
11	-583	-6347	27247
12	-414	-7756	822
13	-208	-8190	-27768
14	35	-7392	-51954
15	315	-5105	-63465
16	632	-1072	-52328
17	986	4964	-6868
18	1377	13260	86292
2109	9214221	847708332	43334279006

$N=19$

X_1	X_2	X_3	X_4
1	-177	60231	-1366698
2	-328	98617	-1855278
3	-453	118013	-1730998
4	-552	121274	-1224263
5	-625	111255	-530625
6	-672	90811	189217
7	-693	62797	809417
8	-688	30068	1238982
9	-657	-4521	1421772
10	-600	-38115	1336500
11	-517	-67859	996732
12	-408	-90898	450887
13	-273	-104377	-217763
14	-112	-105441	-891093
15	75	-91235	-1416125
16	288	-58904	-1605028
17	527	-5593	-1235118
18	792	71553	-48858
19	1083	175389	2246142
2470	5923554	151518164955	29313184373744

$N=20$

X_1	X_2	X_3	X_4
1	-589	74271	-625974
2	-1096	122997	-870264
3	-1521	149333	-841874
4	-1864	156434	-636319
5	-2125	147455	-335625
6	-2304	125551	-8329
7	-2401	93877	290521
8	-2416	55588	519366
9	-2349	13839	650136
10	-2200	-28215	668250
11	-1969	-67419	572616
12	-1656	-100618	375631
13	-1261	-124657	103181
14	-784	-136381	-205359
15	-225	-132635	-497125
16	416	-110264	-705764
17	1139	-66113	-751434
18	1944	2973	-540804
19	2831	100149	32946
20	3800	228570	1090125
2870	75698546	263025418365	6912570985161

$N=21$

X_1	X_2	X_3	X_4
1	-325	9063	-1015683
2	-607	15162	-1442100
3	-846	18644	-1438830
4	-1042	19856	-1146790
5	-1195	19145	-688235
6	-1305	16858	-166758
7	-1372	13342	332710
8	-1396	8944	743900
9	-1377	4011	1019205
10	-1315	-1110	1129680
11	-1210	-6072	1065042
12	-1062	-10528	833670
13	-871	-14131	462605
14	-637	-16534	-2450
15	-360	-17390	-497130
16	-40	-16352	-938408
17	323	-13073	-1224595
18	729	-7206	-1235340
19	1178	1596	-831630
20	1670	13680	-144210
21	2205	29393	1868517
3311	26425091	4446965374	20377674345270

$N=22$

X_1	X_2	X_3	X_4
1	−119	2191	−1353541
2	−223	3699	−1958121
3	−312	4600	−2006590
4	−386	4970	−1670420
5	−445	4885	−1099705
6	−489	4421	− 423161
7	−518	3654	251874
8	−532	2660	839440
9	−531	1515	1274955
10	−515	295	1515215
11	−484	− 924	1538394
12	−438	−2066	1344044
13	−377	−3055	953095
14	−301	−3815	407855
15	−210	−4270	− 227990
16	−104	−4344	− 869376
17	17	−3961	−1409861
18	153	−3045	−1721625
19	304	−1520	−1655470
20	470	690	−1040820
21	651	3661	314279
22	847	7469	2623159
3795	4037880	293611560	40371883933560

$N=23$

X_1	X_2	X_3	X_4
1	− 781	26257	−254309
2	−1468	44695	−374119
3	−2061	56143	−392439
4	−2560	61430	−338800
5	−2965	61385	−239255
6	−3276	56837	−116379
7	−3493	48615	10731
8	−3616	37548	126456
9	−3645	24465	218655
10	−3580	10195	278665
11	−3421	− 4433	301301
12	−3168	−18590	284856
13	−2821	−31447	231101
14	−2380	−42175	145285
15	−1845	−49945	36135
16	−1216	−53928	− 84144
17	− 493	−53295	−199869
18	324	−47217	−291879
19	1235	−34865	−337535
20	2240	−15410	−310720
21	3339	11977	−181839
22	4532	48125	82181
23	5819	93863	517891
4324	197152780	47415244040	1584264017390

Table 2. Coefficients of $X_n^{(N)}(x)$.

	x^4	x^3	x^2	x
$N=2$				
$X_2^{(2)}(x)$			$\frac{5}{2}$	$-\frac{9}{2}$
—		—	—	—
—	—	—	—	—
$N=3$				
$X_2^{(3)}(x)$			7	−18
$X_3^{(3)}(x)$		$\frac{19}{6}$	−14	$\frac{83}{6}$
—	—	—	—	—
$N=4$				
$X_2^{(4)}(x)$			3	−10
$X_3^{(4)}(x)$		$\frac{155}{6}$	−150	$\frac{1171}{6}$
$X_4^{(4)}(x)$	$\frac{23}{8}$	$-\frac{275}{12}$	$\frac{445}{8}$	$-\frac{475}{12}$
$N=5$				
$X_2^{(5)}(x)$			$\frac{11}{2}$	$-\frac{45}{2}$
$X_3^{(5)}(x)$		$\frac{115}{6}$	$-\frac{275}{2}$	$\frac{664}{3}$
$X_4^{(5)}(x)$	$\frac{847}{12}$	−700	$\frac{25355}{12}$	$-\frac{3725}{2}$
$N=6$				
$X_2^{(6)}(x)$			$\frac{13}{2}$	$-\frac{63}{2}$
$X_3^{(6)}(x)$		$\frac{16}{3}$	$-\frac{91}{2}$	$\frac{523}{6}$
$X_4^{(6)}(x)$	$\frac{91}{12}$	$-\frac{539}{6}$	$\frac{3887}{12}$	$-\frac{1022}{3}$
$N=7$				
$X_2^{(7)}(x)$			5	−28
$X_3^{(7)}(x)$		$\frac{17}{6}$	−28	$\frac{373}{6}$
$X_4^{(7)}(x)$	$\frac{413}{12}$	$-\frac{1421}{3}$	$\frac{23815}{12}$	$-\frac{7273}{3}$

	x^4	x^3	x^2	x
$N=8$				
$X_2^{(8)}(x)$			17	-108
$X_3^{(8)}(x)$		$\frac{109}{6}$	-204	$\frac{3089}{6}$
$X_4^{(8)}(x)$	$\frac{595}{12}$	-777	$\frac{44489}{12}$	-5157
$N=9$				
$X_2^{(9)}(x)$			$\frac{19}{2}$	$-\frac{135}{2}$
$X_3^{(9)}(x)$		$\frac{170}{3}$	$-\frac{1425}{2}$	$\frac{12083}{6}$
$X_4^{(9)}(x)$	$\frac{589}{12}$	$-\frac{1725}{2}$	$\frac{55385}{12}$	-7200
$N=10$				
$X_2^{(10)}(x)$			$\frac{7}{2}$	$-\frac{55}{2}$
$X_3^{(10)}(x)$		$\frac{415}{6}$	$-\frac{1925}{2}$	$\frac{9034}{3}$
$X_4^{(10)}(x)$	$\frac{113}{12}$	$-\frac{550}{3}$	$\frac{13045}{12}$	$-\frac{11275}{6}$
$N=11$				
$X_2^{(11)}(x)$			23	-198
$X_3^{(11)}(x)$		$\frac{199}{6}$	-506	$\frac{10415}{6}$
$X_4^{(11)}(x)$	$\frac{23}{8}$	$-\frac{737}{12}$	$\frac{3197}{8}$	$-\frac{9097}{12}$
$N=12$				
$X_2^{(12)}(x)$			25	-234
$X_3^{(12)}(x)$		$\frac{47}{6}$	-130	$\frac{2911}{6}$
$X_4^{(12)}(x)$	$\frac{1855}{24}$	$-\frac{7189}{4}$	$\frac{305525}{24}$	$-\frac{105157}{4}$
$N=13$				
$X_2^{(13)}(x)$			$\frac{9}{2}$	$-\frac{91}{2}$
$X_3^{(13)}(x)$		$\frac{137}{6}$	$-\frac{819}{2}$	$\frac{4955}{3}$
$X_4^{(13)}(x)$	$\frac{777}{4}$	$-\frac{14651}{3}$	$\frac{149613}{4}$	$-\frac{501137}{6}$

	x^4	x^3	x^2	x
$N=14$				
$X_2^{(14)}(x)$			$\frac{29}{2}$	$-\frac{315}{2}$
$X_3^{(14)}(x)$		$\frac{790}{6}$	$-\frac{5075}{2}$	$\frac{65993}{6}$
$X_4^{(14)}(x)$	$\frac{14413}{12}$	$-\frac{64925}{2}$	$\frac{3207545}{12}$	-641725
$N=15$				
$X_2^{(15)}(x)$			31	-360
$X_3^{(15)}(x)$		$\frac{1805}{6}$	-6200	$\frac{172441}{6}$
$X_4^{(15)}(x)$	$\frac{1953}{4}$	$-\frac{42350}{3}$	$\frac{497395}{4}$	$-\frac{958150}{3}$
$N=16$				
$X_2^{(16)}(x)$			11	-136
$X_3^{(16)}(x)$		$\frac{409}{6}$	-1496	$\frac{44309}{6}$
$X_4^{(16)}(x)$	$\frac{605}{12}$	$-\frac{4658}{3}$	$\frac{174823}{12}$	$-\frac{119578}{3}$
$N=17$				
$X_2^{(17)}(x)$			$\frac{35}{2}$	$-\frac{459}{2}$
$X_3^{(17)}(x)$		$\frac{23}{3}$	$-\frac{357}{2}$	$\frac{5609}{6}$
$X_4^{(17)}(x)$	$\frac{103}{12}$	$-\frac{561}{2}$	$\frac{33515}{12}$	-8109
$N=18$				
$X_2^{(18)}(x)$			$\frac{37}{2}$	$-\frac{513}{2}$
$X_3^{(18)}(x)$		$\frac{257}{6}$	$-\frac{2109}{2}$	$\frac{17519}{3}$
$X_4^{(18)}(x)$	$\frac{851}{12}$	-2451	$\frac{309727}{12}$	$-\frac{158517}{2}$
$N=19$				
$X_2^{(19)}(x)$			13	-19
$X_3^{(19)}(x)$		$\frac{2855}{6}$	-12350	$\frac{432631}{6}$
$X_4^{(19)}(x)$	$\frac{34853}{24}$	$-\frac{635075}{12}$	$\frac{14103895}{24}$	$-\frac{22834675}{12}$

	x^4	x^3	x^2	x
$N=20$				
$X_2^{(20)}(x)$			41	-63
$X_3^{(20)}(x)$		$\frac{3155}{6}$	-14350	$\frac{528571}{6}$
$X_4^{(20)}(x)$	$\frac{13489}{24}$	$-\frac{258475}{12}$	$\frac{6036635}{24}$	$-\frac{10278275}{12}$
$N=21$				
$X_2^{(21)}(x)$			$\frac{43}{2}$	$-\frac{693}{2}$
$X_3^{(21)}(x)$		$\frac{347}{6}$	$-\frac{3311}{2}$	$\frac{31982}{3}$
$X_4^{(21)}(x)$	$\frac{9331}{12}$	-31262	$\frac{4595711}{12}$	$-\frac{2736349}{2}$
$N=22$				
$X_2^{(22)}(x)$			$\frac{15}{2}$	$-\frac{253}{2}$
$X_3^{(22)}(x)$		$\frac{38}{3}$	$-\frac{759}{2}$	$\frac{15347}{6}$
$X_4^{(22)}(x)$	$\frac{3563}{4}$	$-\frac{224917}{6}$	$\frac{1922815}{4}$	$\frac{5392948}{3}$
$N=23$				
$X_2^{(23)}(x)$			47	-828
$X_3^{(23)}(x)$		$\frac{829}{6}$	-4324	$\frac{182657}{6}$
$X_4^{(23)}(x)$	$\frac{1739}{12}$	$-\frac{19113}{3}$	$\frac{1024177}{12}$	$-\frac{1000293}{3}$