ACCEPTANCE INSPECTION BY VARIABLES WHEN THE MEASUREMENTS ARE SUBJECT TO ERROR

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The situation considered is that in which measurement of the characteristic of interest is not exact but subject to appreciable error. The error is assumed to be unbiased and independent of the actual value of the characteristic measured. The population and error variances, σ^2 and σ_e^2, are assumed to be such that σ/σ_e has a known lower limit which is greater than zero. The probability distributions involved are assumed to be normal while the actual values and measurement errors each form a random sample. For suitable specified acceptable and unacceptable fractinos defective, and for σ_e assumed known and unknown, this paper presents one-sided acceptance inspection criteria which are optimum in a specified sense, and which have the property that the producer's and consumer's risks have specified upper bounds.

1. Introduction

Acceptance inspection is a procedure for deciding whether an item lot is of acceptable quality on the basis of a sample of items from the lot. Let us suppose that the characteristic of interest is measurable and that an item can be classified as defective or nondefective on the basis of a given upper or lower limit for the value of this characteristic. If only the knowledge of whether an item is defective or nondefective is used, the acceptance inspection is said to be based on attributes. If the values of the measurements themselves are used, the plan is said to be based on variables.

Let us consider some properties of the ordinary type of acceptance inspection plan (attributes or variables). The items inspected are

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considered to be a random sample from an infinite population of items. Let two fractions p_1 and p_2 ($p_1 < p_2$) be specified. The lot is considered to be of acceptable quality if the fraction p of defectives in the item population is p_1 or less. The lot is considered to be of unacceptable quality if the item population has a fraction defective of p_2 or more. The plan is constructed so that the probability of rejecting the lot is at most α when the item population has a fraction defective of at most p_1, and is at least $(1-\beta)$ when the item population has a fraction defective of at least p_2. The quantity α is called the producer's risk while β is called the consumer's risk.

Consider the case in which there is no measurement error. That is, the value of the measurement equals the true value of the characteristic measured. Then a measurement value lies in the defective region if and only if the item is defective. When the measurement is subject to error, however, its value may be in the defective region when item is nondefective, and vice versa. Thus the fraction defective in a sample is not necessarily the same as the fraction defective in the set of corresponding measurements. For the case of acceptance inspection by attributes, this problem of misclassification is discussed in reference [1], pp. 23-24.

When variables are used, the population of interest is the population of true values of the characteristic measured. Hence the problem is to combine measurements containing error is such a way that the requirements of producer's and consumer's risks are satisfied for the population of true values. To reduce the magnitude of the measurement error, several separate measurements of the characteristic of interest may be made for each item of the sample taken from the item lot. This procedure yields item averages which have the same expected values as the individual measurements but smaller measurement errors. For the plans developed in this paper, the same number of measurements are made for each item; call this number m.

The value of a measurement equals the sum of the true value of the characteristic measured and the value of the measurement error. To derive the results of this paper, several assumptions are made concerning the probability properties of the true values and the measurement errors. These assumptions are

1) The measurement errors are statistically independent of each other and of the true values. They form a random sample from a normal population with known zero mean and variance σ_e^2.

2) The true values form a random sample from a normal population with unknown mean μ and unknown variance σ^2. The value of σ/σ_e has a known positive lower limit R and a known upper limit $\overline{R}$ which may be positive or infinite

$$(0<R\leqq\sigma/\sigma_e\leqq\overline{R}<\infty) \quad \text{or} \quad (0<R\leqq\sigma/\sigma_e<\overline{R}=\infty).$$

Let us discuss the implications and practical validity of these assumptions.

The assumption that the true values and the measurement errors form random samples from infinite populations does not seem to be very restrictive. This should be approximately the case if the items selected for measurement are a legitimate sample from the lot and if the items are produced and measured under standardized conditions. The assumption of normal populations is not a severe limitation and is approximated in many practical situations. This point is considered in [2], pp. 49–50, and the discussion will not be repeated here.

If the measuring apparatus is of good quality and trained personnel are used, it is often permissible to assume that the remaining conditions of 1) are satisfied. By accurate calibration, the measurement error can be made very nearly unbiased and almost independent of the magnitude of the measurement.

With respect to 2), conservative values for the constants R and $\overline{R}$ can usually be obtained on the basis of past experience with situations of a similar nature. It is important, however, that the values of R and $\overline{R}$ be chosen without any knowledge of the measured values for the items selected from the lot. Otherwise, the probability properties derived for the criteria presented in this paper might be appreciably changed.

Only one-sided acceptance inspection criteria are developed. That is, only situations are considered in which an item is determined to be defective on the basis of a single limit. There are two cases. In one case an item is classified as defective if and only if the true value of the characteristic measured exceeds a specified value U. In the other case an item is classified as defective if and only if the true value of the characteristic measured is less than a specified value L. Let us dispose of the second case once and for all by replacing each observation by its negative and replacing the lower limit L by the upper limit $U=-L$.

The method used to develop the one-sided criteria for the case of an upper limit is suggested by that used by W. Allen Wallis in [2]. The one-sided acceptance criterion in [2] is of the form

Accept the lot if and only if $\bar{x}+ks \leqq U$.

Here, $\bar{x}$ is the mean of the values obtained for the N items sampled, and s is their standard deviation (computed using $N-1$ in the denominator). This criterion has the property that

$$P\{\text{accept lot} \mid p \leqq p_1\} \geqq 1-\alpha \qquad (1.1)$$
$$P\{\text{reject lot} \mid p \geqq p_2\} \geqq 1-\beta$$

For the case σ_e known, the one-sided criterion presented in this paper is of the form

Accept the lot if and only if $\bar{x}+k_0 s \leqq U+V_0\sigma_e$

For the case σ_e unknown, the criterion is

Accept the lot if and only if $\bar{x}+k_{00} s \leqq U$.

Here, the value for an item is the average of the measurements made on that item. In most parametric situations, the criteria approach the criterion of [2] in the limit, as the number of measurements becomes infinite, see subsection 2.5, and have the following property:

Suppose we may assume that

$$0<R \leqq \sigma/\sigma_e \leqq \bar{R} < \infty \text{ or } 0<R \leqq \sigma/\sigma_e < \bar{R}=\infty.$$

Then, if $\bar{R}$, R, m, p_1, and p_2 satisfy a certain condition (the condition depends on whether we are assuming σ_e known or unknown), criteria of the given form (i.e., $\bar{x}+ks \leqq U+V\sigma_e$ for σ_e known, $\bar{x}+ks \leqq U$ for σ_e unknown) satisfying (1.1) exist, and the recommended test is the one (of all these) requiring the least sample size N. Note that the case of $\bar{R}$ infinite is of course of special interest; in many practical situations (where p_1 and p_2 are not too close, because of restrictions on the size of N), the condition on $\bar{R}$, R, m, p_1, and p_2 will be satisfied for $\bar{R}$ infinite, so that no finite upper bound need be assumed for σ, and this case of special interest will in fact obtain.

To apply the criteria of this paper, it is not necessary to make direct use of $\bar{x}$ and s; the short-cut methods outlined in [2], pp. 33–41 are directly applicable if the value for an item is taken to equal the average of the measurements made on that item.

In the ensuing derivations, we shall assume that $0<p_1<p_2<1/2$, and that

1) N is not too small (say $N\geqq 5$)
2) $\alpha+\beta<1$
3) $\beta\leqq 1/2$ (assumed only for the case σ_e known).

Conditions 1) and 2) correspond to those postulated by Wallis in [2], and are intended to insure the validity of the fundamental normality assumption and the simple algebraic relations among the test parameters resulting therefrom. In the present discussion, the normality assumption is that s, the standard deviation of the item averages, is distributed normally with mean $\sqrt{\sigma^2+\sigma_e^2/m}$ and variance $(\sigma^2+\sigma_e^2/m)/2N$ (with the "rounding upward" provision discussed in [2] on p. 61).

Condition 3) is an added assumption found useful in the present discussion.

2. Statement of results

2.1. *Notation*

Some of the notation has already been introduced. Further notation is as follows:

$$\frac{1}{\sqrt{2\pi}}\int_{K\varepsilon}^{\infty} e^{-\frac{t^2}{2}}dt=\varepsilon$$

$$K_1=K_{p_1},\ K_2=K_{p_2},\ K=K_\alpha+K_\beta$$

$$\overline{S}=\sqrt{\overline{R}^2+m^{-1}},\quad S=\sqrt{R^2+m^{-1}}$$

$$\lambda=K_1R-K_2\overline{R}$$

$$\phi=K_\alpha S+K_\beta\overline{S};\ \text{if } K_\beta\geqq 0,\ \phi>0 \text{ since } \alpha+\beta<1.$$

$$q=\frac{K_2}{m\overline{R}},\quad p=\frac{K_2}{mR},\quad c=\left(\frac{K_2}{K_1}\right)p$$

$$d=(K_1)\left(\frac{\overline{R}S-R\overline{S}}{\overline{S}-S}\right)$$

$$l=\left(\frac{K_1}{mR}\right)\left(1-\sqrt{\left(1-\left(\frac{K_2}{K_1}\right)^2\right)(1+mR^2)}\right)$$

$$f_{K_i,R}(v)=(v+K_iR)/\sqrt{R^2+m^{-1}},\ i:1,2$$

$$g_{K_i}(v)=(mv^2+K_i^2)^{1/2},\ i:1,2$$

$$K_1^*(v)=\begin{cases} f_{K_1,R}(v)\equiv f_1(v) & \text{if}\quad v\leqq d \\ f_{K_1,\bar{R}}(v)\equiv \bar{f}_1(v) & \text{if}\quad v\geqq d \end{cases}$$

$$K_2^*(v)=\begin{cases} f_{K_2,\bar{R}}(v)\equiv \bar{f}_2(v) & \text{if}\quad v\leqq q \\ g_{K_2}(v)\equiv g_2(v) & \text{if}\quad q\leqq v\leqq p \\ f_{K_2,R}(v)\equiv f_2(v) & \text{if}\quad v\geqq p \end{cases}$$

$$\Delta(v)=K_1^*(v)-K_2^*(v)$$

$$\Sigma(v)=K_\alpha K_2^*(v)+K_\beta K_1^*(v)$$

$$k(v)=\frac{\Sigma(v)}{K}$$

$$N(v)=\frac{1}{2}+\frac{2K^2+\Sigma^2(v)}{2\Delta^2(v)}$$

(Note that $N(v)$, as defined, is to be rounded to the *nearest* integer; this corresponds to rounding $N(v)-1/2$ *upward*, as suggested on p. 61 of [2].)

$$v_1=\frac{2KS\bar{S}(\bar{S}-S)}{\lambda\phi}-\frac{(K_\beta K_1R\bar{S}+K_\alpha K_2\bar{R}S)}{\phi}$$

v_2: the unique solution of

$$\left[g_2(v)+\left(\frac{K_\beta}{K}\right)(f_1(v)-g_2(v))\right][vmK_1R-K_2^2]=2[g_2(v)-Smv]$$

in the interval $c<v<p$.

2.2. *The Case σ_e Known*

The situation of principal interest is that in which p_1, p_2, α and β are specified (where, as discussed on p. 7, we assume $p_1<p_2<1/2$, $\alpha+\beta<1$ and $\beta\leq 1/2$).

If it can be assumed that $0<R\leqq\sigma/\sigma_e\leqq\bar{R}<\infty$ or $0<R\leqq\sigma/\sigma_e<\bar{R}=\infty$, then a test satisfying (1.1), with acceptance criterion of type

$$\bar{x}+ks\leqq U+v\sigma_e$$

exists if and only if $l<d$, and the recommended test is given by the

acceptance criterion

$$\bar{x}+k_0 s \leqq U+v_0\sigma_e$$

where v_0 is defined as follows:

$$\begin{array}{ll}
\text{If } \lambda>0 \text{ and } v_1<q,\ v_2\leqq q, & \text{then } v_0=v_1 \\
\text{If } \lambda>0 \text{ and } v_1<q,\ v_2>q, & \text{then } v_0=(v_1, v_2) \\
\text{If } \lambda>0 \text{ and } v_1\geqq q, & \text{then } v_0=\max(q, v_2) \\
\text{If } \lambda=0, & \text{then } v_0=v_2 \\
\text{If } \lambda<0 \text{ and } c\geqq d, & \text{then } v_0=d \\
\text{If } \lambda<0 \text{ and } c<d, & \text{then } v_0=\min(d, v_2)
\end{array}$$

where (,) denotes whichever of the two arguments assigns the smaller value to $N(v)$, and

$$k_0=\sum(v_0)/K\ .$$

The N (number of items required by the test) is given by $\max[5, N(v_0)]$.

Note that the parametric situation $\lambda<0$ will often obtain, in which case

$$v_0=\min(d, v_2)$$

and, if, further, $c\geq d$,

$$v_0=d\ .$$

Within the approximations discussed near the end the Introduction, the recommended test satisfies (1.1), and among all tests satisfying (1.1) with acceptance criterion of form ($\bar{x}+ks\leqq U+v\sigma_e$), requires the smallest N.

The treatment of the other two cases of interest is simply given in terms of the above.

Suppose, for example, that all parameters except β are given, with $l<d$, and that we are trying to ascertain the smallest value of β, call it β_0, that can be achieved by tests of the given type. Consider the function $N_\beta(v_0(\beta))$ obtained by computing $N(v_0)$ for different values of β and fixed values for all test parameters other than N and β. β_0 is then "the" (see subsection J at end of paper) solution of

$$\begin{array}{l}
\beta\leq 1/2 \\
\beta<1-\alpha \\
N_\beta(v_0(\beta))=N\ .
\end{array}$$

If no solution exists, no test of type ($\bar{x}+ks \leq U+v\sigma_e$) with $\beta \leq 1/2, <1-\alpha$, and the given N exists. The search for the solution β_0 is much simplified by the fact that $N_\beta(v_0(\beta))$ is essentially decreasing. The test parameters will be given by:

$$v=v_0(\beta_0)$$
$$k=\frac{K_\alpha K_2^*(v_0(\beta_0))+K_{\beta_0}K_1^*(v_0(\beta_0))}{K_\alpha+K_{\beta_0}}$$

The case when α is not specified is exactly analogous, except that the first condition on the value of α is not imposed, so that the equation determining α_0 is simply

$$\alpha<1-\beta$$
$$N_\alpha(v_0(\alpha))=N.$$

2.3. *The Case σ_e Unknown*

The situation of principal interest again is that for which p_1, p_2, α, and β are specified (where, as discussed at the end of the Introduction, we assume $p_1<p_2<1/2$, $\alpha+\beta<1$).

If it can be assumed that $0<\underline{R}\leqq\sigma/\sigma_e\leqq\overline{R}<\infty$ or $0<\underline{R}\leqq\sigma/\sigma_e<\overline{R}=\infty$, then a test with acceptance criterion of type

$$\bar{x}+ks\leqq U$$

exists if and only if

$$\frac{K_2}{K_1}<\frac{\underline{R}\overline{S}}{\overline{R}\underline{S}}$$

and the recommended test is given by the acceptance criterion

$$\bar{x}+k_{00}s\leqq U$$

where $k_{00}=\sum(0)/K$ and $N=\max[5,\ N(0)]$. The recommended test has the property that, within the approximations discussed at the end of the Introduction, (1.1) is satisfied, and, further, that among all tests of the form $\bar{x}+ks\leqq U$ satisfying (1.1), the given test requires the smallest N.

The treatment of the other two cases again follows readily from the above. Suppose for example that $K_\alpha>0$. Then $N_\beta(0)$ (the function obtained by computing $N(0)$ for different values of β and fixed values for all test parameters others than N and β) is strictly decreasing in

β, for $\beta \leqq 1/2$. Hence, if all test parameters except β are specified, with $K_2/K_1 < R\bar{S}/\bar{R}S$, then β_0, the smallest value of $\beta \leqq 1/2$ that can be achieved by tests of the given type, will be obtained by solving

$$N = \frac{2(K_\alpha + K_{\beta_0})^2 + [K_\alpha K_2^*(0) + K_{\beta_0} K_1^*(0)]^2}{2[K_1^*(0) - K_2^*(0)]^2} + \frac{1}{2} ,$$

and k_{00} will be given by

$$k_{00} = \frac{K_\alpha K_2^*(0) + K_{\beta_0} K_1^*(0)}{K_\alpha + K_{\beta_0}} .$$

2.4. *Properties of $\bar{R}$ and m*

Properties of $\bar{R}$. We note first the important special case $\bar{R} = \infty$ may be obtained from the general results by passage to the limit. For example, in the case σ_e known, the necessary and sufficient condition for the existence of a solution becomes

$$l < d_\infty = K_1(S - R)$$

and, when α and β are given,

$$v_0 = \min(d_\infty, v_2)$$

since, clearly, $\lambda = -\infty < 0$.

It is also important to note that, since

$$\lim_{\bar{R} \to R} d = \frac{K_1}{mR} > l$$

solutions will always exist, for values of $\bar{R}$ sufficiently close to R.

Another observation stems from the development in section I) of the derivations, and illustrates the fact that N is not in general a strictly increasing fuction of $\bar{R}$ in the case σ_e known. For example, let $\bar{R}'$ be defined as in I), and assume that all parameters of the problem are fixed, except $\bar{R}$ and N. Then, if $K_1R < K_2\bar{R}'$, the optimum test and sample size, i.e., v, k, and N, are the same for all $\bar{R}$ with either

$$\frac{K_1R}{K_2} \leqq \bar{R} \leqq \bar{R}'$$

or

$$K_2/v_2m \leqq \bar{R} < K_1R/K_2$$
$$v_1 \geqq q .$$

It follows that, before reluctantly choosing, in the interests of economy, a dangerously low value for $\overline{R}$, the experimenter would do well to ascertain whether the other test parameters may not be such that a larger, safer value can be used without the slightest additional cost.

Determination of m. Consider the problem of deciding how many measurements to make on each item of the sample taken from the lot. The value of m can usually be determined on the basis of obtaining minimum over-all cost for the acceptance inspection. Let p_1, p_2, α, β be specified and R and $\overline{R}$ known. For the criteria of this paper, N will usually decrease as m increases. Thus, increasing m tends to reduce the cost from the viewpoint of the number of items in the sample from the lot. On the other hand, increasing m tends to increase the cost since the number of measurements is increased. This suggests that the over-all expense can be minimized by choosing m in a suitable fashion.

Suppose that a function $C(N, m)$ is given which furnishes the over-all cost of the inspection for given N and m. For the criteria presented in this paper (and given p_1, p_2, α, β, R, $\overline{R}$), the value of N is determined by that of m. Hence, the over-all expense can be considered to be a function of m alone. For the usual practical situation, this cost will be a monotonically increasing function of m or will have a unique minimum. This suggests computing total cost for successive integers until the cost for some $M+1$ is greater than that for M. Then the value M is chosen for m. In many cases, M will equal 1.

2.5. *Properties of the Test Given by Wallis*

The test given by wallis in [2] prescribes the values 0 and

$$k^* = (K_\alpha K_2 + K_\beta K_1)/(K_\alpha + K_\beta)$$

for $v\sigma_e$ and k, assuming that there is no measurement error, and that no bounds are imposed on σ. Since (v_0, k_0) and (k_{00}) represent optimizations, it seems of some interest to ascertain whether, for given values of R and $\overline{R}$,

1) $(v_0, k_0) \rightarrow (0, k^*)$ as $m \rightarrow \infty$.
2) $k_{00} \rightarrow k^*$ as $m \rightarrow \infty$.

It is not hard to verify that 2) holds without exception.

In the case of 1), it turns out that (v_0, k_0) also tends to $(0, k^*)$ in most cases. The exceptions are as follows: Lot $v_{1\infty}$ represent the value of v_1 for m infinite. Then, if $\lambda>0$ and $v_{1\infty}<0$, $v_0 \to v_{1\infty}$, which is not equal to 0. Note that, in the limit, the optimum test depends on R, $\overline{R}$ and σ_e only through $R\sigma_e$ and $\overline{R}\sigma_e$, which may simply be thought of as two bounds, $\underline{\sigma}$ and $\overline{\sigma}$, on σ. Hence, when bounds are assumed for σ, the test given by Wallis can be improved upon when $\lambda>0$ and $\sigma_e v_{1\infty}$ (a function of $R\sigma_e$ and $\overline{R}\sigma_e$, i.e. $\underline{\sigma}$ and $\overline{\sigma}$)<0.

3. Numerical example

Let us illustrate the construction of an acceptance criterion of the form developed in this paper for the case σ_e known, by supposing that we have given

$$p_1=0.01,\ p_2=0.03,\ \alpha=0.05,\ \beta=0.10,\ \sigma_e=1,\ R=1,$$
U arbitrary.

Then (by any convenient table of the normal distribution, for example [2], Table 1.1)

$$K_1=2.32635,\ K_2=1.88079,\ K_\alpha=1.64485,\ K_\beta=1.28155.$$

In order to illustrate the savings in sample size that may be accomplished by assuming a finite value for $\overline{R}$ or by measuring each item in the sample more than once, we shall consider six combinations of $\overline{R}$ and m:

$$(\overline{R}, m)=(\infty, 1),\ (\infty, 2),\ (\infty, 3),\ (4, 1),\ (4, 2),\ (4, 3).$$

For these six cases the major steps in the calculation of the criterion are shown in Table 1. In all six cases we verify that $l<d$; $\lambda<0$; $d<c$. Therefore, for all six cases, a solution exists and $v_0=d$; the value v_0 is then used in computing k_0 and N. The values of N shown in Table 1 should be rounded to the nearest integer (see p. 8), giving sample sizes of 648, 305, 236, 429, 260, and 214 respectively for the six cases.

The reduction in sample size from 648 to 429, from 305 to 260, or from 236 to 214, illustrates the saving that can be accomlpished by assuming a finite value for $\overline{R}$. As might be anticipated, this advantage diminishes as the number of measurements per item increases. The reduction from 648 through 305 to 236, or from 429 through 260 to 214, illustrates the saving that can be accomplished by measuring each item more than

Table 1
Numerical Example

	$\bar{R}=\infty$			$\bar{R}=4$		
	$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
l	.39011	−.02253	−.13730	.39011	−.02253	−.13730
d	.96360	.52282	.35989	1.31714	.68624	.46527
c	1.52057	.76028	.50686	1.52057	.76028	.50686
$K_1{}^*(v_0)$	2.3263	2.3263	2.3263	2.5763	2.4598	2.4176
$K_2{}^*(v_0)$	2.11327	2.02090	1.98140	2.29613	2.11642	2.04617
k_0	2.20658	2.15466	2.13246	2.41884	2.26679	2.20883
N	648.3	305.4	235.6	428.6	259.8	213.5

Note : all symbols are explained in subsection 2.1.

once. As discussed in subsection 2.5, if the number of measurements per item were increased indefinitely, N would tend to the sample size required for the test given by Wallis [2], p. 23, i.e., $N=137$.

If testing is nondestructive, so that the items in the sample may be returned to the lot after inspection, it would perhaps be reasonable to assume that the over-all cost of inspection is proportional to the total number of meaurements made, i.e., that

$$C(N, m)=wmN,$$

where w is the cost of making a single measurement. Then, in the above example if $\bar{R}=\infty$, $C(N, m)$ would be minimized by taking $m=2$; but, if $\bar{R}=4$, $C(N, m)$ would be minimized by taking $m=1$.

It is also interesting to investigate how much N is increased by assuming no value for σ_e. As an illustration, we pick the values $(\infty, 3)$ and $(4, 3)$ for $(\bar{R}, m)$. We note first that the condition

$$\frac{K_2}{K_1}<\frac{R\bar{S}}{\bar{R}S}$$

is satisfied in both instances. The values for $(K_1^*(0), K_2^*(0))$ are, respectively, (2.01468, 1.88079) and (2.01468, 1.86150). Hence N equals 1,377 and 1,044, as compared with 236 and 214 for σ_e assumed known and equal to 1.

4. Derivations

4.1. *Derivation of the Case σ_e Known*

We begin with the situation of primary interest, that corresponding to given α and β; the solution for the other two situations will then be seen to follow readily. For clarity, the ensuing development is divided into several parts.

A) We introduce a coordinate system in the plane, the "x-axis" measuring $(\mu-U)/\sigma_e$, the "y-axis" measuring $[(\sigma/\sigma_e)^2+m^{-1}]^{1/2}$. Then, to any parameter set (μ, σ, σ_e), there corresponds a point, namely

$$\left(\frac{\mu-U}{\sigma_e},\ [(\sigma/\sigma_e)^2+m^{-1}]^{1/2}\right),$$

in the half-plane $y \geqq m^{-1/2}$ (which we denote henceforth by P); conversely, to any point (x, y) in P, there correspond populations (μ, σ) and measurement distributions (σ_e) with $(\mu-U)/\sigma_e=x$ and $[(\sigma/\sigma_e)^2+m^{-1}]^{1/2}=y$. We next observe that the locus in our (x, y) plane of all points corresponding to populations (μ, σ) satisfying an inequality of type

$$\frac{U+v\sigma_2-\mu}{\sqrt{\sigma^2+\sigma_e^2/m}}\begin{Bmatrix}\geqq K_1^*\\ \leqq K_2^*\end{Bmatrix},\quad \text{or}\quad \frac{v-x}{y}\begin{Bmatrix}\geqq K_1^*\\ \leqq K_2^*\end{Bmatrix};\quad K_1^*, K_2^*>0$$

is that part $\begin{Bmatrix}R_1(v)\\ R_2(v)\end{Bmatrix}$ of P which includes all points lying either $\begin{Bmatrix}\text{below}\\ \text{above}\end{Bmatrix}$ or on the line $\begin{Bmatrix}l_1(v)\\ l_2(v)\end{Bmatrix}$, of slope $\begin{Bmatrix}-K_1^{*-1}\\ -K_2^{*-1}\end{Bmatrix}$, passing through the point $(v, 0)$ and $\begin{Bmatrix}0,\ v/K_1^*)\\ 0,\ v/K_2^*)\end{Bmatrix}$.

We also determine the locus $\begin{Bmatrix}r_1\\ r_2\end{Bmatrix}$ of all points in P corresponding to populations (μ, σ) satisfying an inequality of type

$$\frac{U-\mu}{\sigma}\begin{Bmatrix}\geqq K_1\\ \leqq K_2\end{Bmatrix}\ \text{or}\ \frac{-x}{\sqrt{y^2-m^{-1}}}\begin{Bmatrix}\geqq K_1\\ \leqq K_2\end{Bmatrix};\ K_1, K_2>0.$$

It is readily verified that, for $x \leqq 0$, $\begin{Bmatrix}r_1\\ r_2\end{Bmatrix}$ consists of points lying on or $\begin{Bmatrix}\text{below}\\ \text{above}\end{Bmatrix}$ the upper branch $\begin{Bmatrix}b_1\\ b_2\end{Bmatrix}$ of the hyperbola $\begin{Bmatrix}h_1\\ h_2\end{Bmatrix}$, h_i having center $(0, 0)$, major axis $2m^{-1/2}$, minor axis $\begin{Bmatrix}2K_1m^{-1/2}\\ 2K_2m^{-1/2}\end{Bmatrix}$, and equation

$$\frac{y^2}{m^{-1}}-\frac{x^2}{K_i^2m^{-1}}=1,\quad i:\ 1, 2.$$

Note that h_i becomes slimmer as K_i becomes small, ranging from the lines $y=\pm m^{-1/2}$ for $K_i=\infty$ to part of the line $x=0$ for $K_i=0$.

The region r_1 contains no points with $x>0$, so that it is precisely the region characterizd above, that bounded by the line $y=m^{-1/2}$ and the curve b_1, and for which $x\leqq 0$. On the other hand, r_2, in addition to the region described for $x\leqq 0$, further contains all points in P with $x\geqq 0$.

Suppose now that we are given two regions in P, r_1 and r_2, of type described above, determined by two hyperbolas, h_1 and h_2 (h_2 "slimmer" than h_1), with common center at (0, 0), common major axis $2m^{-1/2}$, common vertical transverse axis, and corresponding to the two hypothesis regions $\frac{U-\mu}{\sigma}\geqq K_1$ and $\frac{U-\mu}{\sigma}\leqq K_2$. Consider pairs of negatively sloped straight lines, $(l_1(v), l_2(v))$, issuing from common points $(v, 0)$ on the x-axis and determining pairs of regions $(R_1(v), R_2(v))$ in P, of the type described above, such that:

Condition A: The slope of $l_2(v)$ has greater absolute value than that of $l_1(v)$.

Condition B: $R_1(v)$ includes r_1.

Condition C: $R_2(v)$ includes r_2.

It is not hard to verify that a test (v, k, N) satisfying (1.1), with acceptance criterion

$$\frac{U+v\sigma_e-\bar{x}}{S}\geqq k$$

exists if and only if a line pair $(l_1(v), l_2(v))$ exists, which satisfies conditions A, B, and C. As a matter of fact, there is a straightforward one-to-one correspondence between tests (v, k, N) of the given type satisfying (1.1), and line pairs of type $(l_1(v), l_2(v))$. This correspondence is as follows: Let K_1^*, K_2^* be the negative inverses of the slopes of $l_1(v)$ and $l_2(v)$. Then

$$\begin{aligned} v&=v \\ K_1^*&=\frac{K_\alpha}{\sqrt{N}}\sqrt{1+k^2/2}+k \qquad (4.1) \\ K_2^*&=\frac{-K_\beta}{\sqrt{N}}\sqrt{1+k^2/2}+k\ . \end{aligned}$$

It further follows, therefore, that the search for the best test

(v_0, k_0, N_0) satisfying (1.1) is equivalent to selecting that line pair $(l_1^0(v), l_2^0(v))$, with slopes $(-K_{10}^{*-1}, -K_{20}^{*-1})$, from among those satisfying conditions A, B, and C, which in addition satisfies

Condition D:

$$N(K_{10}^*, K_{20}^*) = \frac{1}{2} + \frac{2(K_\alpha + K_\beta)^2 + (K_\alpha K_{20}^* + K_\beta K_{10}^*)^2}{2(K_{10}^* - K_{20}^*)^2}$$

$$\leqq N(K_1^*, K_2^*),\ (K_1^*, K_2^*) \text{ corresponding to any}$$

$$(l_1(v), l_2(v)) \text{ satisfying conditions A, B and C.}$$

It is clear, first of all, that no line pair satisfying A, B, and C exists at all, unless we restrict attention to that part of P given by $y \geqq S$ $(\sigma/\sigma_e \geqq R)$. Retricting attention to this part of P means that we require A, B, and C to be satisfied only for those parts of R_1, r_1, R_2, and r_2 lying on or above the line $y=S$. Besides delimiting the problem in this manner, we may also wish to assume an upper bound $\overline{R}\sigma_e$ for σ, i.e., restrict attention to that part of P given by $S \leqq y \leqq \overline{S}$.

We now investigate in detail the necessary and sufficient conditions for the existence of at least one line pair satisfying A, B, and C in the strip $S \leqq y \leqq \overline{S}$, and the method of selecting the pair satisfying D, if pairs satisfying A, B, and C do exist. Let us note, once and for all, that passage to the limit $\overline{R} = \infty$ in the ensuing formulas yields the important special case in which no upper bound $\overline{R}\sigma_e$ for σ is introduced. We begin with some definitions:

P_i (i=1, 2): the intersection point (in the second quadrant) of the line $y=S$ with h_i.

Q_i $(i=1, 2)$: the intersection point (in the second quarant) of the line $y=\overline{S}$ with h_i.

d: the number such that the lines P_1Q_1 and $y=0$ intersect at $(d, 0)$.

q: the number such that the line through $(q, 0)$ and Q_2 is tangent to h_2 at Q_2.

p: the number such that the line through $(p, 0)$ and P_2 is tangent to h_2 at P_2.

$\begin{Bmatrix} L_1(v) \\ L_2(v) \end{Bmatrix}$: the set of all negatively sloped lines passing through $(v, 0)$ and satisfying condition $\begin{Bmatrix} B \\ C \end{Bmatrix}$.

$\begin{Bmatrix}\mathscr{L}_1(v)\\ \mathscr{L}_2(v)\end{Bmatrix}$: the line in $\begin{Bmatrix}L_1(v)\\ L_2(v)\end{Bmatrix}$ whose slope has $\begin{Bmatrix}\text{smallest}\\ \text{largest}\end{Bmatrix}$ absolute value.

$\begin{Bmatrix}K_1^*(v)\\ K_2^*(v)\end{Bmatrix}:\begin{Bmatrix}-1/S_1(v)\\ -1/S_2(v)\end{Bmatrix}$, where $S_i(v)$ is the slope of $\mathscr{L}_i(v)$.

It is easily verified that d, p, and q are as given by the formulas of subsection 2.1. We next investigate the nature of $\mathscr{L}_1(v)$ and $\mathscr{L}_2(v)$.

From inspection of the geometric figures involved it is clear that, for $v\leqq d$, $\mathscr{L}_1(v)$ is the line through $(v, 0)$ and P_1, while, for $v\geqq d$, $\mathscr{L}_1(v)$ is the line through $(v, 0)$ and Q_1. Further, for $v\leqq q$, $\mathscr{L}_2(v)$ is the line through $(v, 0)$ and Q_2; for $q\leqq v\leqq p$, $\mathscr{L}_2(v)$ is the line through $(v, 0)$ which is tangent to the branch b_2; and, for $v\geqq p$, $\mathscr{L}_2(v)$ is the line through $(v, 0)$ and P_2. Computing the expressions for the negative reciprocals of the slopes of $\mathscr{L}_1(v)$ and $\mathscr{L}_2(v)$, we obtain the formulas for $K_1^*(v)$ and $K_2^*(v)$ of subsection 2.1.

B) We establish now that, assuming

$$0<\underline{R}\leqq\sigma/\sigma_e\leqq\overline{R}<\infty \quad\text{or}\quad 0<\underline{R}\leqq\sigma/\sigma_e<\overline{R}=\infty,$$

a test of the required form satisfying (1.1) exists if and only if

$$\text{(4.2A)}\qquad \sqrt{1-\left(\frac{K_2}{K_1}\right)^2}>\frac{(\sqrt{m})(\overline{S}S-\overline{R}R-m^{-1})}{\overline{S}-S}$$

or, equivalently,

$$\text{(4.2B)}\qquad l<d\ .$$

To show this, we recall that a test of required form satisfying (1.1) exists if and only if a line pair $(l_1(v), l_2(v))$ exists satisfying conditions A, B, and C. It is clear that such a pair will exist if and only if $\Delta(v)\equiv K_1^*(v)-K_2^*(v)>0$ for some v.

It is easily verified that either $q<p\leqq d$ or $q<d<p$. If

$$q<p\leqq d$$

it follows straightforwardly from the definitions that

$$\text{(4.3)}\qquad \begin{array}{lll} \text{for}\ \ v\leqq q, & \Delta(v)=f_1(v)-\overline{f}_2(v) & \text{is increasing}\\ \text{for}\ \ q\leqq v<p, & \Delta(v)=f_1(v)-g_2(v) & \text{is increasing}\\ \text{for}\ \ p\leqq v\leqq d, & \Delta(v)=f_1(v)-f_2(v) & \text{is constant}\\ \text{for}\ \ d\leqq v, & \Delta(v)=\overline{f}_1(v)-f_2(v) & \text{is decreasing} \end{array}$$

and if

$$q<d<p$$

it also follows immediately that

$$\text{(4.4)} \qquad \begin{array}{lll} \text{for } v \leqq q, & \Delta(v)=f_1(v)-\bar{f}_2(v) & \text{is increasing} \\ \text{for } q \leqq v \leqq d, & \Delta(v)=f_1(v)-g_2(v) & \text{is increasing} \\ \text{for } d \leqq v \leqq p, & \Delta(v)=\bar{f}_1(v)-g_2(v) & \text{is decreasing} \\ \text{for } p \leqq v, & \Delta(v)=\bar{f}_1(v)-f_2(v) & \text{is decreasing.} \end{array}$$

Suppose now that $q<p\leqq d$; then it is readily checked that $\Delta(v)>0$ for any v with $p\leqq v\leqq d$, so that $\Delta(v)>0$ for some v.

If, on the other hand, $q<d<p$, then, by (4.4), $[\Delta(v)>0$ for some $v]$ if and only if $[\Delta(d)>0]$ or, equivalently, if and only if $[l<d]$, and it is readily checked that $[l<d]$ may be written in the form given by (4.2A).

But

$$\frac{K_2}{K_1}>1-\sqrt{(1+mR^2)(1-(K_2/K_1)^2)}$$

so that

$$[p\leqq d] \text{ implies } [(4.2)].$$

Hence it follows that $\Delta(v)>0$ for some v if and only if (4.2) holds.

C) Next, we establish that $N(v)$ is an increasing function of v for $v\geqq\min(d, p)$, $\Delta(v)>0$. This is done by writing $N(v)$ as a function of $K_2^*(v)$ and $\Delta(v)$, which is readily seen to be increasing in $K_2^*(v)$ and decreasing in $\Delta(v)$ for $K_3\geqq 0$, $\Delta(v)>0$. Since $K_2^*(v)$ is an increasing function of v and $\Delta(v)$ is a nonincreasing function of v for $v\geqq\min(d, p)$ by A) above, the desired property of $N(v)$ follows.

D) We now examine the behavior of the function

$$N_1(v)=\frac{1}{2}+\frac{2K^2+\sum_1^2(v)}{2\Delta_1^2(v)}$$

where $\Delta_1(v)=f_1(v)-\bar{f}_2(v)$ and $\sum_1(v)=K\cdot\bar{f}_2(v)+K\cdot f_1(v)$, in the region defined by $\Delta_1(v)>0$. We note first that this region is given by:

$$v>\frac{(K_2\bar{R}S-K_1R\bar{S})}{S-S}\equiv e\,.$$

We further show that:

1) if $\lambda>0$, then $e<v_1$ and

$$\begin{aligned} N_1'(v)&<0 \quad \text{for} \quad e<v<v_1 \\ N_1'(v)&=0 \quad \text{for} \quad v=v_1 \\ N_1'(v)&>0 \quad \text{for} \quad v>v_1 \end{aligned}$$

2) if $\lambda\leqq 0$, $N_1'(v)<0$ for all $v>e$.

For convenience, we drop the subscript 1, and note first that

$$N'(v)=\frac{\Delta^2\sum\sum'-[\sum^2+2K^2]\Delta\Delta'}{\Delta^4}.$$

For $v>e$, Δ and Δ' are greater than 0. Hence,

$$N'(v)\gtreqqless 0 \quad \text{if} \quad \frac{\Delta}{\Delta'}\cdot\sum\sum'-\sum^2-2K^2\gtreqqless 0.$$

Since ϕ, $\sum'\neq 0$ (note that $\phi=\sum'\cdot S\bar{S}$), let $\sum/\sum'=v-f$ and note that $\Delta/\Delta'=v-e$. Hence $N'\gtreqqless 0$ for $v>e$ if

$$(v-f)(f-e)\gtreqqless 2\left(\frac{K}{\sum'}\right)^2. \tag{4.5}$$

But $f-e=S\bar{S}K\lambda(\bar{S}-S)^{-1}\phi^{-1}$. Hence, for $v>e$,

$$\text{if} \quad \lambda=0, \; N'<0.$$

Further, if $\lambda>0$, $e<f<v_1$, where v_1 is the solution of (4.5) with equality, so that $f-e>0$ and, for $v>e$;

$$N'\gtreqqless 0 \quad \text{if} \quad v\gtreqqless v_1>e$$

while, if $\lambda<0$, $e>f>v_1$, so that $f-e<0$ and $N'\gtreqqless 0$ if $v\gtreqqless v_1$; or, since $e>v_1$, $N<0$ for all $v>e$.

E) We next examine the behavior of the function

$$N_2(v)=\frac{1}{2}+\frac{2K^2+\sum_2^2(v)}{2\Delta_2^2(v)}$$

where

$$\begin{aligned} \Delta_2(v)&=f_1(v)-g_2(v) \\ \sum\nolimits_2(v)&=K_\alpha g_2(v)+K_\beta f_1(v), \end{aligned}$$

in the region defined by $\Delta_2(v)>0$, $\Delta_2'(v)\geqq 0$. We note first that this region given by:

$$p\equiv\frac{K_2}{mR}\geqq v>\left(\frac{K_1}{mR}\right)\left(1-\sqrt{(1+mR^2)\left(1-\left(\frac{K_2}{K_1}\right)^2\right)}\right)\equiv l.$$

We further show that, for $l<v\leqq p$, $N_2'(v)\gtreqless 0$ if $v\gtreqless v_2$, where $l<c<v_2<p$, $c=\frac{K_2 p}{K_1}$, and v_2 is the unique solution of

$$\left[g_2(v)+\frac{K_\beta \Delta_2(v)}{K}\right][vmK_1R-K_2^2]=2[g_2(v)-Smv]$$

in the interval $(c<v<p)$.

For convenience, we drop the subscript 2, and note that, since $\Delta(v)>0$ for $l<v\leqq p$, $N'\gtreqless 0$ for $l<v\leqq p$ if $\Delta\sum\sum'-[\sum^2+2K^2]\Delta'\gtreqless 0$, or, rewriting this condition, if

$$[f_1(v)-g_2(v)][\textstyle\sum(v)]\left[\frac{K_\beta}{S}+\frac{K_\alpha mv}{g_2(v)}\right]-[\textstyle\sum^2(v)+2K^2]\left[\frac{1}{S}+\frac{mv}{g_2(v)}\right]\gtreqless 0$$

or if

$$\left[g_2(v)+\frac{K_\beta}{K}\Delta(v)\right][K_1Rmv-K_2^2]\gtreqless 2[g_2(v)-Smv]\,.$$

Now write this last condition as

$$F_1(v)F_2(v)\gtreqless F_3(v)\ .$$

Since K_β is assumed $\geqq 0$ and K is assumed >0, $F_1(v)$ is nondecreasing for $l<v\leqq p$ by A). Further, $F_2(v)$ is an increasing, and $F_3(v)$ a decreasing, function of v. We notice next that, for $l<v\leqq p$, $F_1(v)$ has no root, $F_2(v)$ has exactly one root (at $v=c$), and $F_3(v)$ has exacly one root (at $v=p$). It follows that F_1F_2-F_3 is increasing for $l<v\leqq p$, and is equal to zero for $v=v_2$, where $c<v_2<p$. Hence our assertion concerning the behavior of N_2'.

F) We also need the following:

1) $[\lambda\gtreqless 0]$ is equivalent to $[c\gtreqless q]$
2) $[\lambda\geqq 0]$ implies $[p<d]$
3) $e<l$, except when $e=l=q$, and $[l<q]$ is equivalent to $[e<q]$.

G) We are now ready to describe the location of the value v_0 of v minimizing $N(v)$, in the region given by $\Delta(v)=K_1^*(v)-K_2^*(v)>0$, or $\bar{v}>v>\underline{v}$, where $\underline{v}=l$ when $l\geqq q$, and $\underline{v}=e<q$ when $l<q$. The upper bound $\bar{v}$ imposes no effective limitation, since, by the argument in B) and by C), we need look only at v's less than or equal to $\min(d, p)$. We also recall that $N(v)=N_1(v)$ for $v\leqq q$, and $N(v)=N_2(v)$ for $v\geqq q$.

It will be useful to distinguish three parametric situations:

1) $\lambda > 0$.

By D), E), and F), we have: $p<d$; $c<q$ (hence $l<q$ and $\underline{v}=e<q$); $N_1'(v) \gtreqless 0$ for $v>e$, if $v \gtreqless v_1 > e < q$; $N_2'(v) \gtreqless 0$ for $l<v \leqq p$, if $v \gtreqless v_2$, where $c<v_2<p$.

Hence, $N_1(v)$ may have its unique minimum for $v>e$ at v_1, $e<v_1<q$, or at $v_1 \geqq q$; and $N_2(v)$ may have its unique minimum for $l<v\leqq p$ at $v_2 \leqq q$ or at v_2, $q<v_2\leqq p$.

Hence,

$$\begin{array}{lllll} \text{if} & v_1<q & \text{and} & v_2\leqq q, & v_0=v_1 \\ \text{if} & v_1<q & \text{and} & v_2>q, & v_0=(v_1, v_2) \\ \text{if} & v_1\geqq q, & & & v_0=\max(q, v_2) \end{array}$$

where $(\,,)$ denotes whichever of the two arguments assigns the smaller value to $N(v)$.

2) $\lambda = 0$.

By D), E), and F), we have: $p<d$, $c=q$ (hence $l<q$ and $\underline{v}=e<q$); $N_1'(v)<0$ for all $v>e$; $N_2'(v)\gtreqless 0$ for $l<v\leqq p$, if $v\gtreqless v_2$ where $q=c<v_2<p$.

Hence, $N_1(v)$ is decreasing for $e<v\leqq q$ and $N_2(v)$ must have its unique minimum for $l<v\leqq p$ at v_2, $q<v_2<p$.

Hence,

$v_0=v_2$.

3) $\lambda<0$.

By D), E), and F), we have: $e\leqq l$, $c>q$, $N_1'(v)<0$ for all $v>e$; $N_2'(v)\gtreqless 0$ for $l<v\leq p$, if $v\gtreqless v_2$, where $q<c<v_2<p$.

Hence, if $e<q$, then $\underline{v}=e$ and $N_1'(v)<0$ for $e<v\leqq q$, while, if $e\geqq q$, then $l\geqq q$ and $\underline{v}=l$, so that, in either case, v_0 cannot lie outside $[q, p]$. Further, $N_2(v)$ must have its unique minimum for $l<v\leqq p$ at $v_2, q<c<v_2<p$.

Hence

$v_0=\min(d, v_2)$.

Further, since $v_2>c$, if $c>d$, then

$$v_0=d.$$

H) In summary, then, if it can be assumed that

$$0<\underline{R}\leqq\sigma/\sigma_e\leqq\overline{R}<\infty \quad \text{or} \quad 0<\underline{R}\leqq\sigma/\sigma_e<\overline{R}=\infty,$$

a test with acceptance criterion of the form

$$\bar{x}+ks \leqq U+v\sigma_e$$

will exist if and only if

$$l<d\ .$$

Further, of all possible tests of this form, the one minimizing N is given by:

$$v=v_0,\ v_0 \text{ as given in G)},$$
$$k=k(v_0)=\frac{K_\alpha K_2^*(v_0)+K_\beta K_1^*(v_0)}{K_\alpha+K_\beta}\ ,$$

and will require a sample size of

$$N(v_0)=\max\left[5,\ \frac{1}{2}+\frac{2K^2+\sum^2(v_0)}{2\Delta^2(v_0)}\right].$$

I) It is worthwhile to note that, for fixed values of the other parameters, N is not in general a strictly increasing function of $\bar{R}$:

1) By G), for $\lambda=0$, $v_0=v_2$; for $\lambda<0$, $v_0=\min(d, v_2)$, and, for $\lambda>0$, $v_0=\max(q, v_2)$, if $v_1 \geqq q$.

2) v_2 does not depend on $\bar{R}$, and, also, since

$$c<v_2,\ \ K_2/v_2m<K_1R/K_2.$$

3) d, considered as a function of $\bar{R}$, i.e. $d(\bar{R})$, is monotone strictly decreasing in $\bar{R}$, with $d(R)=K_1/mR$; also, $q(\bar{R})$ is of course strictly decreasing in $\bar{R}$, with

$$q\left(\frac{K_2}{v_2m}\right)=v_2 \quad \text{and} \quad q\left(\frac{K_1R}{K_2}\right)=c.$$

Now define $\bar{R}'$ by

$$d(\bar{R}')=v_2$$

and confine attention to situations for which

$$\frac{K_1R}{K_2}<\bar{R}'.$$

Then for all $\bar{R}$ with

$$\frac{K_1R}{K_2}\leqq\bar{R}\leqq\bar{R}'$$

we will hve

1) $d>l$ (since, for all such $\bar{R}$, we have: $d(\bar{R})\geqq d(\bar{R}')=v_2>l$)

so that a test of the required form exists, and

2) $v_0=v_2$

Further, for all $\bar{R}$ with

$$\frac{K_2}{v_2 m}\leqq\bar{R}<\frac{K_1 R}{K_2}$$

and

$$v_1\geqq q$$

we have:

1) $d>l$ (since $d(\bar{R})>d(\bar{R}'))$ and

2) $v_0=v_2$.

Hence, in summary, if $K_1R<K_2\bar{R}'$, then for all $\bar{R}$ with

$K_1R/K_2\leqq\bar{R}\leqq\bar{R}'$ or with $K_2/v_2m\leq\bar{R}<K_1R/K_2$ and $v_1\geqq q$, $v_0=v_2$, $k=k(v_2)$ and $N=N(v_2)$, hence is *constant*.

J) The solution for the other two possible stuations is simply given in terms of the above. Suppose, for example, that N and α are fixed and $l<d$, and that we are trying to ascertain the smallest value of β, call it β_0, that can be achieved by tests of the given type. Consider the function $N_\beta(v_0(\beta))$ of β, i.e., the function of β obtained by computing $N(v_0)$ for different values of β and the same fiixed values for all test parameters other than β and N, β_0, if it exists at all, will be the smallest solution of

$$\beta\leqq\frac{1}{2}$$

$$\beta<1-\alpha$$

$$N_\beta(v_0(\beta))=N\,.$$

Note that, as is to be expected, $N_\beta(v_0(\beta))$ will be strictly decreasing in β in almost every case (clearly so, for example, for ranges of β for which $v_0=d$ or q). Hence, "smallest solution" may, for practical purposes, be replaced by "solution" v and k will be given by

$$v=v_0(\beta_0)$$

$$k=\frac{K_\alpha K_1^*(v_0(\beta_0))+K_{\beta_0}K_2^*(v_0(\beta_0))}{K_\alpha+K_{\beta_0}}\,.$$

The solution for α unspecified is analogous.

4.2. *The Case of σ_e Unknown*

The development for the case σ_e unknown is simpler than and analogous to that for σ_e known. The reason for restricting attention to tests (v, k, N) with $v \equiv 0$ lies in the fact that, when σ_e is not kown, the correspondence of equations (4.1) is one-to-one for no other values of v.

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