NOTE ON LINEAR PROGRAMMING

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1. Introduction

In this paper, we treat some computational methods of linear programming. As well-known, any linear programming problems can be completely solved by the simplex method proposed by G. B. Dantzig. Of course, some modified, or short-cut methods have been so far studied for special linear programming problems, for example, transportation problems, production scheduling problems, etc. However, they are all based upon the same principle as the simplex method.

The recent development of automatic computer with large capacity of memory makes it possible to solve linear programming problems of considerably large scale which is not manageable for hand computer or electric computer. The methods treated in this paper seem to be useful for solving of linear programming problem by automatic computer.

Suppose x^0 be an optimal solution of the following linear programming problem;

Maximize $c'x$

subject to

(1.1) $$Ax \leqq P_0, \quad x \geqq 0$$

where $A=(a_{ij}) \quad i=1, 2, \cdot\cdot h, \quad m; j=1, 2, \cdots, n$

$P_0'=(b_1, b_2, \cdots, b_m), \quad c'=(c_1, c_2, \cdots, c_n)$

$x'=(x_1, x_2, \cdots, x_n), \quad x^{0\prime}=(x_1^0, x_2^0, \cdots, x_n^0)$.

Let the optimal solution x^0 be already known by computation.

When the coefficient matrix A, the vector b and c are changed, we obtain a new problem. We discuss in this paper whether there is any method which enables us to solve the new problem by using the information that the optimal solution of the original problem is x^0. The reason why we discuss such a question is that the use of the above information seems to make it easy or economic to solve the new problem.

In section 2, we treat the case where b-vector is changed. In this case, the basis corresponding to x^0 in the original problem is not always the one corresponding to the optimal solution of the new problem.

In section 3, the case, where the matrix A is changed, is treated. In section 4, we treat the case where c-vector, the coefficient vector of the objective function, varies. This case is very easy to treat.

In section 5, we treat the new problem which is derived from the original problem by adding some restrictions, and in section 6 the new problem derived from the original problem by elimination of some restrictions.

2. Change of P_0-vector.

Let A be $(m \times n)$ rectangular matrix, and $P_1, P_2, \cdots, P_n$ be its column vectors. Then, without loss of generality, the original problem (1.1) can be written in the following form;

Maximize:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{2.1}$$

subject to

$$P_1x_1 + P_2x_2 + \cdots + P_nx_n = P_0 \tag{2.2}$$
$$x_i \geqq 0 \quad (i=1, 2, \cdots, n).$$

Now, let the optimal solution x^0 of this problem be known by "Simplex" method, and let the optimal basis corresponding to x^0 be $P_1, P_2, \cdots, P_m$. (The last assumption does not put any restriction on us, for we are always possible to satisfy this assumption by relabeling the suffices of P_i.)

Therefore, the following relation is satisfied;

$$P_1x_1^0 + P_2x_2^0 + \cdots + P_mx_m^0 = P_0 \tag{2.3}$$

$$x_i^0 > 0 \quad (i=1, 2, \cdots, m). \tag{2.4}$$

Let us now consider the new problem where P_0 is changed into $P_0+\varepsilon$, that is, the new problem is to maximize

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$P_1x_1 + P_2x_2 + \cdots + P_nx_n = P_0 + \varepsilon \tag{2.5}$$
$$x_i \geqq 0 \quad (i=1, 2, \cdots, n)$$

where $\varepsilon' = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)$.

Now, let B be the square matrix whose column vectors are $P_1, P_2,$

$\cdots$, P_{m-1} and P_m, and let B^{-1} be the inverse matrix of B.

Because $\{P_1, P_2, \cdots, P_m\}$ is the optimal basis of the original problem, B^{-1} is already known from its optimal tableau.
Suppose β_i $(i=1, 2, \cdots, m)$ be the i-th row vector of B^{-1}, that is,

$$B^{-1}=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}.$$

Then, we obtain

$$\varepsilon=\sum_{i=1}^{m}(\beta_i\varepsilon)P_i \tag{2.6}$$

Therefore, we obtain from (2.3) and (2.6)

$$P_0+\varepsilon=\sum_{i=1}^{m}(x_1^0+\beta_i\varepsilon)P_i\,.$$

We must here consider the following two cases.

i) $x_i^0+\beta_i\varepsilon\geqq 0 \quad (i=1, 2, \cdots, m)$.

ii) There exists at least negative one among $x_1^0+\beta_i\varepsilon$ $(i=1, \cdots, m)$

Case i).

In this case, $\{P_1, P_2. \cdots, P_m\}$ is obviously a feaasible basis. Further, it is easily proved that it is an optimal basis for the simplex tableau of the new problem with the basis $\{P_1, P_2, \cdots, P_m\}$ coincides with the optimal one of the original problem except the P_0-column, which assures that the optimality criterion is satisfied, that is, $Z_j-c_j\geqq 0$ $(j=1, 2, \cdots, n)$.

c	$\longrightarrow$		c_1	c_2	$\cdots$	c_n	c_{m+1}	$\cdots$	c_n
$\downarrow$		$P_0+\varepsilon$	P_1	P_2,	$\cdots$	P_m	P_{m+1}	$\cdots$	P_n
c_1	P_1	$x_1^0+\beta_1\varepsilon$							
c_2	P_2	$x_2^0+\beta_2\varepsilon$			$x_{ij}\begin{pmatrix}i=1, 2, \cdots, m\\ j=1, 2, \cdots, n\end{pmatrix}$				
$\vdots$	$\vdots$	$\vdots$							
c_m	P_m	$x_m^0+\beta_m\varepsilon$							
	Z_i-c_j		Z_1-c_1	Z_2-c_2,	$\cdots$	Z_n-c_m,		$\cdots$	Z_n-c_n

Fig. 2.1.

Therefore, the optimal solution is

$$x_i=x_i^0+\beta_i\varepsilon \qquad (i=1, 2, \cdots, m)$$
$$x_j=0 \qquad (j=m+1, \cdots, n)$$

Case ii).

In this case, $\{P_1, P_2, \cdots, P_m\}$ is not a feasible basis of the new problem. However, we can obtain the optimal solutive of the new problem by the modification of the problem. Let $x^0_{i_k}+\beta_{i_k}\varepsilon$ $(k=1, 2, \cdots, l)$ be all of negative ones among $x^0_i+\beta_i\varepsilon$ $(i=1, 2, \cdots, m)$.

Consider the following modified problem; Maximize objection function:

$$\sum_{j=1}^{n} c_j x_j - M\sum_{k=1}^{l} y_{i_k} \tag{2.7}$$

subject to

$$\sum_{j=1}^{n} P_j x_j + \sum_{k=1}^{l}(-P_{i_k})y_{i_k} = P_0+\varepsilon \tag{2.8}$$

$$x_j \geqq 0 \ (j=1, 2, \cdots, n),\ y_{i_k}\geqq 0 \ (k=1, 2, \cdots, l)$$

where, M is a sufficiently large positive number.

It is easily shown that, if there exists the optimal solution of the problem (2.5), it coincides with the optimal one of the modified problem, By introducing new vectors $-P_{i_k}$ $(k=1, 2, \cdots, l)$, the following relation is satisfied;

$$\sum_{i=1}^{m} P_i(x^0_i+\beta_i\varepsilon) + \sum_{k=1}^{l}(-P_{i_k})(-x^0_{i_k}-\beta_{i_k}\varepsilon) = P_0+\varepsilon \tag{2.9}$$
$$(i\neq i_k,\ k=1, 2, \cdots, l)$$

$$x^0_i+\beta_i\varepsilon\geqq 0\,;\quad -x^0_{i_k}-\beta_{i_k}\varepsilon\geqq 0\,,\quad k=1, 2, \cdots, l\,,\quad (i\neq i_k).$$

This means that

$$x_i=\delta_i(x^0_i+\beta_i\varepsilon)\,,\qquad \delta_i=\begin{cases}1 & i\neq i_k\\ -1 & i=i_k\end{cases}\ \begin{pmatrix} i=1, 2, \cdots, m\\ k=1, 2, \cdots, l\end{pmatrix}$$
$$x_j=0 \qquad (j=m+1, \cdots, n)$$

is a feasible solution of the modified problem, or, equivalently, $\{\delta_i P_i\}$ $(i=1, 2, \cdots, m)$ is a feasible basis of the modified problem.

c	$\longrightarrow$		c_1	c_2	$\cdots$	c_{i_k}	$\cdots$	c_m	c_{m+1}	$\cdots$	c_n	$-M$
$\downarrow$	Basis	$P_0+\varepsilon$	P_1	P_2	$\cdots$	P_{i_k}	$\cdots$	P_m	P_{m+1}	$\cdots$	P_n	$-P_{i_k}$
c_1	P_1	$x_1{}^0+\beta_1\varepsilon$	....		....		....		....		....	
$\vdots$	$\vdots$	$\vdots$										.
$-M$	$-P_{i_k}$	$-x_{i_k}{}^0-\beta_{i_k}\varepsilon$	....		...	-1			....		....	1
$\vdots$	$\vdots$	$\vdots$										
c_n	P_m	$x_m{}^0+\beta_m\varepsilon$	....		....		....		....		....	

Fig. 2.2.

Figure 2.2 indicates the simplex tableau of the modified problem with the basis $\{\delta_i P_i\}$.

If we apply the "Simplex" method regarding the tableau of Fig. 2.2 as the initial one of the modified problem, we can reach the optimal solution after finite stages of renewing of tableau. Of course, we cannot know previously this number of stages.
But, roughly speaking, it depends on the number of the negative ones among $x_i+\beta_i\varepsilon$ $(i=1, 2, \cdots, m)$. Therefore, it depends on the magnitudes of components of ε.

For the actual computation, the last columns corresponding to $-P_{i_k}$ $(k=1, 2, \cdots, l)$ are not necessary to compute.

3. Change of the Coefficient Matrix A.

In this section, let us consider the case where the coefficient matrix A is changed into $\tilde{A}$.

As in § 2, suppose $A=\{P_1, P_2, \cdots, P_n\}$ and the original linear programming problem be,

Maximize

$$\sum_{j=1}^{n} c_j x_j \tag{3.1}$$

subject to

$$\sum_{j=1}^{n} P_j x_j = P_0 \tag{3.2}$$

$$x_j \geqq 0 \quad (j=1, 2, \cdots, n)$$

For the simplicity of discussion, we treat the case where only one column vector P_1 is changed into $\tilde{P}_1$. Therefore, $\tilde{A}=\{\tilde{P}_1, P_2, \cdots, P_n\}$, that is, the new problem is as follows;

Maximize (3.1)

subject to

$$\tilde{P}_1 x_1 + \sum_{j=2}^{n} P_j x_j = P_0 \tag{3.3}$$

$$x_j \geqq 0 \quad (j=1, 2, \cdots, n).$$

As before, we assume that the optimal solution of the original problem is x^0 and the corresponding optimal basis is B. We want to know an extreme point of the convex set defined by (3.3) from the information about x^0 and B. If we know it, we can take it as the initial point of the simplex method.

Here, it seems convenient to treat separately the following two cases according to whether B contain P_1 or not.

Case (i) $B \ni P_1$

In this case, we can assume $x^{0\prime}=(x_1^0, x_2^0, \cdots, x_m^0, 0 \cdots 0)$ and $B=\{P_1, P_2, \cdots, P_m\}$, without loss of generality.

Case (ii) $B \not\ni P_1$

In this case, we can assume $x^{0\prime}=(0, x_2^0, x_3^0, \cdots, x_{m+1}^0, 0 \cdots 0)$ and $B=\{P_2, P_3, \cdots, P_{m+1}\}$.

In both cases, we can assume $B^{-1}=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}$ without any confusion.

Case (i)

It is obvious that if there exists an optimal solution of the problem (3.3), it coincides with the optimal one of the following modified problem;

Maximize $\qquad -Mx_1+c_1\tilde{x}_1+\sum_{j=2}^{n} c_j x_j$

subject to

$$P_1x_1+P_1\tilde{x}+\tilde{P}_2x_2+\cdots+P_nx_n=P_0$$
$$x_j \geqq 0, \quad \tilde{x}_1 \geqq 0 \quad (j=1, 2, \cdots, n)$$

where M is a sufficiently large positive number as in the previous section.

$\tilde{P}_1$ is represented as the linear combination of vectors in B as follows;

$$\tilde{P}_1=\sum_{i=1}^{m}(\beta_i\tilde{P}_1)P_i=BB^{-1}\tilde{P}_1 .$$

Obviously, B is a feasible basis of the modified problem. Therefore, if we take B as basis, the simplex tableau of the modified problem is as follows;

c	$\longrightarrow$		$-M$	c_1	c_2	c_j	c_m	$\cdots$	c_n
	Basis	P_0	P_1	$\tilde{P}_1$	P_2	P_j	P_m	$\cdots$	P_n
$-M$	P_1	x_1^0							
c_2	P_2	x_2^0		$B^{-1}\tilde{P}_1$					
$\vdots$	$\vdots$	$\vdots$							
c_m	P_m	x_m^0							

Fig. 3.1.

In this way, we have obtained the initial tableau of the modified problem. By application of the simplex method to it, we can reach the optimal solution after some stages of renewing of tableau.

Case (ii)

Consider the same modified problem as in Case (i). Obviously, B is a feasible basis of the modified problem, and $\tilde{P}_1$ is represented as the linear combination of the column vectors of B, that is,

$$\tilde{P}_1=\sum_{i=2}^{m+1}(\beta_i\tilde{P}_1)P_i=BB^{-1}\tilde{P}_1\,.$$

If we take B as basis, the simplex tableau of the modified problem is as follows;

c	$\longrightarrow$		$-M$	c_1'	c_2	$\cdots$	c_n
$\downarrow$	Basis	P_0	P_1	$\tilde{P}_1$	P_2	$\cdots$	P_n
c_2 c_3 $\vdots$ c_{m+1}	P_2 P_3 $\vdots$ P_{m+1}	x_2^0 x_3^0 $\vdots$ x_{m+1}^0		$B^{-1}\tilde{P}_1$			
Z_j-c_j				$*$			

$*=(c_2,\ c_3,\ \cdots,\ c_{m+1})B^{-1}\tilde{P}_1-c_1$.

Because B is the optimal basis of the original problem,

$$z_j-c_j\geqq 0\,,\ \text{for } j=1,\ 2,\ \cdots,\ n\,.$$

Therefore, we treat the following two cases separately

(a) $$(c_2,\ c_3,\ \cdots,\ c_{m+1})B_0^{-1}\tilde{P}_1-c_1\geqq 0$$

(b) $$(c_2,\ c_3,\ \cdots,\ c_{m+1})B^{-1}\tilde{P}_1-c_1<0\,.$$

Case (a)

In this case, it is obvious that the above tableau is an optimal tableau and $(0,\ x_2^0,\ x_3^0,\ \cdots,\ x_{m+1}^0,\ 0,\ \cdots,\ 0)$ is the optimal solution.

Case (b)

We must repeat the renewing of tableau according to the simplex method, until the optimality is satisfied. It will be reached after one or several stages of renewing.

4. Change in the Coefficient Vector of the Objective Function.

In this section, we treat the case where only the coefficients of the objective function are changed and the restriction is the same as the original one. Therefore, let the new objective function be

$$c_1'x_1+c_2'x_2+\cdots+c_n'x_n\,. \tag{4.1}$$

This problem is very simple to treat, if the optimal solution or the optimal basis of the original problem is known. At first, compute

$$z_j-c_j'=(c_1',\ c_2',\ \cdots,\ c_m')B^{-1}P_j-c_j' \qquad (j=1,\ 2,\ \cdots,\ n)$$

where the optimal basis of the original problem is $B=\{P_1, P_2, \cdots, P_m\}$, and B^{-1} is its inverse matrix.

Thus we can get a simplex tableau of the new problem with the basis B.

c			c_1'	c_2'	......	c_n'
↓	Basis	P_0	P_1	P_2	......	P_n
c_1' ⋮ c_m	P_1 ⋮ P_m	x_1^0 ⋮ x_m^0		$B^{-1}A$		
z_j-c_j'			.....	z_j-c_j'	.....	

Fig. 4.1.

If $z_j-c_j'\geqq 0$ $(j=1,\ 2,\ \cdots,\ n)$, then B is the optimal basis of the new problem and the optimal solution is the same as the one of the original problem. If there is at least one negative among z_j-c_j' $(j=1,\ 2,\ \cdots,\ n)$, we must repeat the renewing of simplex tableau until the optimal solution is obtained.

For actual computation, it is serious question how many stages of renewing are necessary until the optimal solution is reached. Of course, it is impossible to know previously the exact number of stages. But the number of the negative among z_j-c_j' $(j=1,\ 2,\ \cdots,\ n)$ may be used as a rough estimation of the number of stages. If the change $c\to c'$ is not so great, the number of stages is also not so great. In actual problems, the change in coefficient usually not so great. Therefore the above method seems to be useful.

5. Addition of Several Restrictions.

Suppose the following linear programming problem is already solved;

Maximize $c'x$,

subject to

Problem (A)
$$Ax\leqq P_0$$
$$x\geqq 0$$

where, $A=\{P_1, P_2, \cdots, P_n\}$, $x'=(x_1, x_2, \cdots, x_n)$, $c'=(c_1, c_2, \cdots, c_n)$.

In this section, we discuss the new problem which is derived when several further restrictions are added to the above problem. For simplicity of illustration, we consider only one restriction to be added.

Let it be the following linear inequality

$$\alpha' x \leqq b_{m+1} \tag{5.1}$$

where $\alpha'=(a_{m+1,1}, a_{m+1,2} \cdots, a_{m+1,n})$.

The restriction of the problem (A) is written by using slack variables as follows;

$$\sum_{j=1}^{n} P_j x_j + \sum_{i=1}^{m} \lambda_i x_{n+i} = P_0 \tag{5.2}$$

$$x_j,\ x_{n+i} \geqq 0 \qquad (j=1, 2, \cdots, n;\ i=1, 2, \cdots, m),$$

where $\lambda_i' = (\underbrace{\overbrace{0, 0, \cdots, 0}^{i-1}, 1, 0, \cdots, 0}_{m})$, $i=1, 2, \cdots, m$.

Suppose the optimal basis be $B=\{P_1, P_2, \cdots, P_m\}$ and suppose the following tableau be the optimal one of the problem (A).

Basis	$\longrightarrow$		c_1 c_2 ..... c_n	0 0 0
$\downarrow$	Basis	P_0	P_1 P_2 ..... P_n	λ_1 λ_2 ..... λ_m
c_1	P_1	x_1^0		
c_2	P_2	x_2^0	$B^{-1}A$	B^{-1}
$\vdots$	$\vdots$	$\vdots$		
c_m	P_m	x_m^0		

Fig. 5.1.

For the formulation of the new problem, we define following $(m+1)$ dimensional vectors.

$$\begin{pmatrix} P_j \\ a_{m+1,j} \end{pmatrix} = \tilde{P}_j, \quad \begin{pmatrix} P_0 \\ b_{m+1} \end{pmatrix} = \tilde{P}_0, \quad \begin{pmatrix} \lambda_i \\ 0 \end{pmatrix} = \tilde{\lambda}_i, \quad \lambda_{m+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}^m$$

$$(j=1, 2, \cdots, n;\ i=1, 2, \cdots, m)$$

Then, the new problem is written as follows;

Maximize $c'x$

subject to

Problem (B) $$\sum_{j=1}^{n} \tilde{P}_j x_j + \sum_{i=1}^{m+1} \tilde{\lambda}_i x_{n+i} = \tilde{P}_0$$

$$x_j \geqq 0, \quad x_{n+i} \geqq 0 \quad (j=1, 2, \cdots, n;\ i=1, 2, \cdots, m+1)$$

If we choose as a basis of problem (B) $\tilde{B}=\{\tilde{P}_1\ \tilde{P}_2, \cdots, \tilde{P}_m, \lambda_{m+1}\}$, the inverse matrix of $\tilde{B}$ is easily obtained because B^{-1} is already known from the optimal tableau of the original problem (A), that is,

$$\tilde{B}^{-1}=\left[\begin{array}{c|c} B & 0 \\ \hline \alpha & 1 \end{array}\right]^{-1}=\left[\begin{array}{c|c} B^{-1} & 0 \\ \hline * & 1 \end{array}\right],$$

where $* = -\alpha' B^{-1}$.

Therefore,

$$\tilde{P}_j=\tilde{B}\tilde{B}^{-1}\tilde{P}_j=\tilde{B}\begin{pmatrix} B^{-1}P_j \\ -\alpha' B^{-1}P_j+a_{m+1,j} \end{pmatrix}\begin{matrix}\}m \\ \}1\end{matrix} \qquad (j=0,\ 1,\ 2,\ \cdots,\ n).$$

Thus, we obtain a simplex tableau of problem which is shown in Fig. 5.2,

c	$\longrightarrow$		c_1 c_2 .... c_n	0 0 0	0
$\downarrow$	Basis	$\tilde{P}_0$	$\tilde{P}_1$ $\tilde{P}_2$ $\tilde{P}_j$ ... $\tilde{P}_n$	$\tilde{\lambda}_1$ $\tilde{\lambda}_2$ ..... $\tilde{\lambda}_m$	λ_{m+1}
c_1	$\tilde{P}_1$	x_1^0			0
c_2	$\tilde{P}_2$	x_2^0	$B^{-1}P_j$	B^{-1}	$\vdots$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
c_m	$\tilde{P}_m$	x_m^0			0
0	$\tilde{\lambda}_{m+1}$	*	**	$-\alpha' B^{-1}$	1

Fig. 5.2.

where $* = -\alpha' B^{-1}P_0+b_{m+1}$, $** = \{-\alpha' B^{-1}P_j+a_{n+1,j};\ j=1,\ 2,\ \cdots,\ n\}$.

Here, we must consider the following two cases; Case (1) $-\alpha' B^{-1}P^0+b_{m+1}\geqq 0$, Case (2) $-\alpha' B^{-1}P_0+b_{m+1}<0$. Case (1). In this case, the basis $\tilde{B}$ is a feasible one of the new problem (B) and further the optimality is satisfied because the row vector $\{Z_j-c_j,\ (j=1,\ 2,\ \cdots,\ n)\}$ is just the the same as the one of the optimal tableau of the original problem.

Case (2). In this case, the basis $\tilde{B}$ is not feasible. Therefore, we treat the modified problem of the new problem.

Maximize

Problem (C) $$\sum_{j=1}^{n} c_j x_j - M x_{n+m+2}$$

subject to

$$\sum_{j=1}^{n} \tilde{P}_j x_j + \sum_{i=1}^{m} \tilde{\lambda}_i x_{n+i} + (-\tilde{\lambda}_{m+1}) x_{n+m+1} = \tilde{P}_0$$

$$x_j \geqq 0,\quad (j=1,\ 2,\ \cdots,\ n+m+1)$$

where M is a sufficiently large positive number.

Then, the optimal solution of this modified problem must agree with the one of the new problem.

If we put $\tilde{B}^*=[\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_m, -\tilde{\lambda}_{m+1}]$, $\tilde{B}^*$ is clearly a feasible basis of the modified problem. Thus, we obtain the following simplex tableau with the basis $\tilde{B}^*$.

c	$\longrightarrow$		$c_1 \quad c_2 \quad \ldots \quad c_j \quad c_n$	$0 \quad 0 \quad \ldots \quad 0$	0	$-M$
$\downarrow$	Basis	P_0	$P_1 \quad P_2 \quad \ldots \quad P_j \quad P_n$	$\tilde{\lambda}_1 \quad \tilde{\lambda}_2 \quad \ldots \quad \tilde{\lambda}_m$	$\tilde{\lambda}_{m+1}$	$-\tilde{\lambda}_{m+1}$
c_1 ⋮ c_m	$\tilde{P}_1$ $\tilde{P}_2$ ⋮ $\tilde{P}_m$		$B^{-1}P_j$	B^{-1}	0	0
$-M$	$-\tilde{\lambda}_{m+1}$	$-*$	$-**$	αB^{-1}	-1	1

$-*=\alpha' B^{-1}P_0-b_{m+1}$ $\qquad -**=\{\alpha' B^{-1}P^j-\alpha_{m+1,j};\ j=1, 2, \ldots, n\}$

Fig. 5.3.

Then we obtain the optimal tableau by applying the simplex method to this tableau. The existence of this optimal solution follows from the existence of the optimal solution of the new problem (B).

The following lemma will be a help to treat the above problem and the related problems already stated in proceding sections.

LEMMA. *Let $\tilde{P}_0, \tilde{P}_1, \cdots, \tilde{P}_N$ be $(m+1)$-dimensional vectors. Suppose there exists a pair of independent vector $\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_{m+1}$ such that*

(5.3) $$\tilde{P}_0=\sum_{i=1}^{m+1}\lambda_i \tilde{P}_i$$

where $\lambda_i \geqq 0$ $(i=1, 2, \cdots, m)$ and $\lambda_{m+1}<0$, and further suppose that

(5.4) $$\tilde{P}_j=\sum_{i=1}^{m+1}\lambda_{ij}\tilde{P}_j \qquad (j=1, 2, \cdots, N)$$

and $$\lambda_{i,m+1}\geqq 0 .$$

Then $\tilde{P}_0$ can not be represented by any positive linear combination of independent $m+1$ vectors among $\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_N$.

Proof) Suppose $\tilde{P}_0$ be the positive linear combination of P_{k_j} $(j=1, 2, \cdots, m+1)$, that is,

(5.5) $$\tilde{P}_0=\sum_{j=1}^{m+1}\lambda_{k_j}\tilde{P}_{k_j}, \quad \lambda_{k_j}>0 \quad (j=1, 2, \cdots, m+1).$$

Then

$$\tilde{P}_0=\sum_{j=1}^{m+1}\lambda_{k_j}\tilde{P}_{k_j}=\sum_{j=1}^{m+1}\lambda_{k_j}\sum_{i=1}^{m+1}\lambda_{ik_j}\tilde{P}_i$$

Thus,

$$\tilde{P}_0=\sum_{i=1}^{m+1}\left(\sum_{j=1}^{m+1}\lambda_{k_j}\lambda_{ik_j}\right)\tilde{P}_i\,. \tag{5.6}$$

From (5.3) and (5.6), we get

$$0>\lambda_{m+1}=\sum_{j=1}^{m+1}\lambda_{k_j}\lambda_{m+1,k_j}\geqq 0$$

This is a contradiction. Thus the lemma has proved. The above lemma assures that, if $\alpha' B^{-1}P_0-b_{m+1}>0$, both $\{\alpha' B^{-1}P_j-a_{m+1,j}\,;\ j=1,\,2,\,\cdots,\,n\}$ and $\alpha' B^{-1}$ can not be non-positive vectors.

6. Elimination of Restriction.

Let the original problem be as follows,

Maximize $c'x$

subject to

$$\begin{aligned} Ax&\leqq P_0\\ \alpha' x&\leqq b_{m+1}\end{aligned} \tag{6.1}$$

By the same notation as in the section 5, we rewrite the above restriction as follows,

$$\sum_{j=1}^{n}\tilde{P}_j\tilde{x}_j+\sum_{i=1}^{m+1}\tilde{\lambda}_i\tilde{x}_{n+i}=\tilde{P}_0\,. \tag{6.2}$$

Under the condition that the optimal solution of the original problem, is $\tilde{x}^0$ we ask for the optimal solution of the new problem obtained by elimination of the last restriction of (6.1), that is,

Maximize $c'x$

subject to

$$Ax\leqq P_0\,. \tag{6.3}$$

We assume that there exists the optimal solution of the new problem. Here, we consider separately two cases according to whether the optimal basis B of the original problem contains the vector $\tilde{\lambda}_{m+1}$ or not.

Case (i) $B\ni\tilde{\lambda}_{m+1}$.

In this case, we can assume that the optimal basis of the original problem is $\{\tilde{P}_1,\,\tilde{P}_2,\,\cdots,\,\tilde{P}_m,\,\tilde{\lambda}_{m+1}\}$.

Then,

$$\sum_{i=1}^{m}\tilde{P}_i\tilde{x}_i^0+\tilde{\lambda}_{m+1}\tilde{x}_{n+m+1}^0=\tilde{P}_0 \tag{6.4}$$

where $\tilde{x}^{0\prime}=(\tilde{x}_1^0, \tilde{x}_2^0, \cdots, \tilde{x}_{n+m+1}^0)$.

From (6.4), we obtain the following relation;

$$\sum_{i=1}^{m} P_i x_i^0 = P_0 \tag{6.5}$$

where, $x^{0\prime}=(\tilde{x}_1^0, \tilde{x}_2^0, \cdots, \tilde{x}_m^0, \overbrace{0, \cdots, 0}^{n})$.

Therefore, x^0 is a feasible solution of the new problem. The optimality of x^0 is easily followed from the fact that $\{\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_m, \tilde{\lambda}_{m+1}\}$ is the optimal basis of the original problem.

Case (ii) $B \not\ni \lambda_{m+1}$.

Without loss of generality, we can assume that the optimal basis of the original problem is $B=\{\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_{m+1}\}$. Let the optimal tableau be as follows;

c	$\longrightarrow$		c_1 c_2 ..... c_j c_n	
$\downarrow$	Basis	$\tilde{P}_0$	$\tilde{P}_1$ $\tilde{P}_2$ ..... $\tilde{P}_j$ $\tilde{P}_n$	$\tilde{\lambda}_1$ $\tilde{\lambda}_2$ ..... $\tilde{\lambda}_{m+1}$
c_1 c_2 $\vdots$ c_{m+1}	$\tilde{P}_1$ $\tilde{P}_2$ $\vdots$ $\tilde{P}_{m+1}$	$B^{-1}P_0$	$B^{-1}\tilde{P}_j$	B^{-1}

Fig. 6.1.

where $B^{-1}=(b_{ij})$, $i, j=1, 2, \cdots, m+1$.

At first, suppose $\sum_{i=1}^{m+1} c_i b_{i,m+1}>0$, then there exists at least one negative among $b_{i,m+1}$ $(i=1, 2, \cdots, m+1)$, which is assured by the coming lemma.

Choose ε_{i_0} such that

$$\max_{b_{i,m+1}<0} \frac{(B^{-1}\tilde{P}_0)_i}{b_{i,m+1}} = \frac{(B^{-1}\tilde{P}_0)_{i_0}}{b_{i_0,m+1}} = -\varepsilon_{i_0} \tag{6.6}$$

Here, without loss of generality we can assume that $i_0=m+1$.

If we put

$$\tilde{\tilde{P}}_0' = \tilde{P}_0' + (0, 0, \cdots, \varepsilon_{m+1}),$$

the following relation holds;

$$B^{-1}\tilde{\tilde{P}}_0 \geqq 0 \tag{6.7}$$

and especially the last element of the vector $B^{-1}\tilde{\tilde{P}}_0$ is zero.

Now, replacing $\tilde{P}_{m+1}$ in the optimal basis by $\tilde{\lambda}_{m+1}$ and considering

$\tilde{\tilde{P}}_0$ instead of $\tilde{P}_0$, the following relation holds, that is,

(6.8) $$\sum_{i=1}^{m} \tilde{P}_i \tilde{\tilde{x}}_i^0 + \tilde{\lambda}_{m+1} 0 = \tilde{\tilde{P}}_0$$

where $\tilde{\tilde{x}}_i^0 = \tilde{x}_i^0 \geqq 0\,,\quad i=1,\ 2,\ \cdots,\ m,\ m+1.$

$\tilde{\tilde{x}}_j^0 = \tilde{x}_j^0 = 0 \quad j=m+2,\ \cdots,\ n+m+1.$

And the simplex tableau of this modified problem with the basis $\{\tilde{P}_1, \cdots, \tilde{P}_m, \tilde{\lambda}_{m+1}\}$ is as follows,

c	$\longrightarrow$		$c_1 \quad c_2 \quad \cdots\cdots \quad c_n$	$0 \quad \cdots\cdots \quad 0$	0
$\downarrow$	Basis	$\tilde{\tilde{P}}_0$	$\tilde{P}_1 \quad \tilde{P}_2 \quad \cdots\cdots \quad \tilde{P}_n$	$\tilde{\lambda}_1 \quad \ldots 1. \quad \tilde{\lambda}_m$	$\tilde{\lambda}_{m+1}$
c_1 $\vdots$ c_m	$\tilde{P}_1$ $\vdots$ $\tilde{P}_m$	$B^{*-1}P_0$	$B^{*-1}P_j$	B^{*-1}	0
0	$\tilde{\lambda}_{m+1}$	0			1

Fig. 6.2.

where $B^{*-1} = (b_{ij}^*),\ i, j=1,\ 2,\ \cdots,\ m$ and

$$b_{ij}^* = b_{ij} - \frac{b_{m+1,j} b_{i,m+1}}{b_{m+1,m+1}}\,.$$

From (6.8), we obtain easily

(6.9) $$\sum_{i=1}^{m} P_i \tilde{\tilde{x}}_i^0 = P_0\,.$$

Thus, $\tilde{x}^{0\prime} = (\tilde{x}_1^0,\ \tilde{x}_2^0,\ \cdots,\ \tilde{x}_m^0,\ \overbrace{0,\ \cdots,\ 0}^{n})$ is a feasible solution of the new problem and the simplex tableau is easily obtained from the above tableau. This is presented in Fig. 6.3.

c	$\longrightarrow$		$c_1 \quad c_2 \quad \cdots\cdots \quad c_n$	$0 \quad \cdots\cdots \quad 0$
$\downarrow$	Basis	P_0	$P_1 \quad P_2 \quad \cdots\cdots \quad P_n$	$\lambda_1 \quad \cdots\cdots \quad \lambda_m$
c_1 c_2 $\vdots$ c_m	P_1 P_2 $\vdots$ P_m	$B^{*-1}P_0$	$B^{*-1}P_j$	B^{*-1}

Fig. 6.3.

Applying the simplex procedure to this tableau, we can reach the optimal solution which we ask for. Especially, if $(B^{-1}\tilde{P}_j)_{m+1} \geqq 0$; $j=1,$ $2, \cdots, n$, and $(B^{-1}\tilde{\lambda}_i)_{m+1} \geqq 0$, $i=1, 2, \cdots, m$, the above tableau is the optimal one, which is proved in the following.

If we note that

$$\tilde{P}_j=\sum_{i=1}^{m}\left\{(B^{-1}\tilde{P}_j)_i-\frac{(B^{-1}\tilde{P}_j)_{m+1}b_{i,m+1}}{b_{m+1,m+1}}\right\}\tilde{P}_i+\frac{(B^{-1}\tilde{P}_j)_{m+1}}{b_{m+1,m+1}}\tilde{\lambda}_{m+1}$$

$$(j=1,\ 2,\ \cdots,\ n)$$

$$\tilde{\lambda}_k=\sum_{i=1}^{m}\left(b_{ik}-\frac{b_{m+1,k}b_{i,m+1}}{b_{m+1,m+1}}\right)\tilde{P}_i+\frac{b_{m+1,k}}{b_{m+1,m+1}}\tilde{\lambda}_{m+1} \tag{6.10}$$

$$(k=1,\ 2,\ \cdots,\ m+1)$$

the increments $\Delta(z_j-c_j)$ in z_j-c_j by replacing $\tilde{P}_{m+1}$ and $\tilde{P}_0$ by $\tilde{\lambda}_{m+1}$ and $\tilde{\tilde{P}}_0$ are

$$\Delta(z_j-c_j)=+c_{m+1}(B^{-1}\tilde{P}_j)_{m+1}-\frac{(\tilde{B}\tilde{P}_j)_{m+1}}{b_{m+1,m+1}}\sum_{i=1}^{m}c_ib_{i,m+1} \tag{6.11}$$

$$=-\frac{(B^{-1}\tilde{P}_j)_{m+1}}{b_{m+1,m+1}}\sum_{i=1}^{m+1}c_ib_{i,m+1}\,.$$

From the optimality of the tableau in Fig. 6.1, we get

$$\sum_{i=1}^{m+1}c_ib_{i,m+1}\geqq 0\ ,$$

and, from the assumption put previously,

$$b_{m+1,m+1}<0\ .$$

Therefore, if $(B^{-1}\tilde{P}_j)_{m+1}\geqq 0$; $j=1,\ 2,\ \cdots,\ n$, and $(B^{-1}\tilde{\lambda}_j)_{m+1}\geqq 0$; $i=1,\ 2,\ \cdots,\ m$, we obtain,

$$\Delta(z_j-c_j)\geqq 0\ ,\quad j=1,\ 2,\ \cdots,\ n+m\ .$$

This means that the optimally criterion is satisfied in our tableau of Fig. 6.3.

Hitherto, we have assumed the existence of at least one negative among $b_{i,m+1}$;$i=1,\ 2,\ \cdots,\ m+1$. But, it is shown that that is always satisfied under the assumption that the new problem has the optimal solution and $\sum_{i=1}^{m+1}c_ib_{i,m+1}>0$.

Lemma. Let the original problem be ;

Maximize $c'x$, subject to $Ax\leqq P_0$, $\alpha'x\leqq b_{m+1}$, $x\geqq 0$, and let the new problem be ;

Maximize $c'x$, subject to $Ax\leqq P_0$, $x\geqq 0$.

If there exist both optimal solutions of the original and the new problem and $\sum_{i=1}^{m+1}c_ib_{i,m+1}>0$ is satisfied in the optimal tableau of the *original problem, then there exists at least one negative among elements of the column vector under λ_{m+1} in the optimal tableau of the original problem.*

Proof) Suppose $b_{i,m+1} \geqq 0$; $i=1, 2, \cdots, m+1$.

For arbitrary positive number M, we define

$$\tilde{\tilde{P}}'_0 = \tilde{P}'_0 + (0, 0, \cdots, 0, M).$$

Then,

$$(6.12) \qquad \tilde{\tilde{P}}_0 = \sum_{i=1}^{m+1} (B^{-1}\tilde{P}_0)_i \tilde{P}_i + M \sum_{i=1}^{m+1} b_{i\ m+1} \tilde{P}_i$$
$$= \sum_{i=1}^{m+1} \{(B^{-1}\tilde{P}_0)_i + M b_{i,m+1}\} \tilde{P}_i,$$

and
$$\tilde{x}_i = (B^{-1}\tilde{P}_0)_i + M b_{i,m+1} \geqq 0;$$

where $\{\tilde{P}_i;\ i=1, 2, \cdots, m+1\}$ is the optimal basis of the original problem.

From (6.12), this is a feasible basis of the following problem;

maximixe $\quad c'x$

subject to

$$(6.13) \qquad Ax \leqq P_0$$
$$\alpha' x \leqq b_{m+1} + M, \quad x \leqq 0$$

and the value of the objective function is

$$(6.14) \qquad \sum c_i \tilde{x}_i = \sum_{i=1}^{m+1} c_i (B^{-1}\tilde{P}_0)_i + M \sum_{i=1}^{m+1} c_i b_{i,m+1}.$$

Because $\sum_{i=1}^{m+1} c_i b_{i,m+1} > 0$ from the assumptin, the value of the objective function increases when M is increased. On the other hand, when M is increased sufficiently, the last restriction is reduced to an inefficient one, that is, for such M, a vector, which satisfies $Ax \leqq P_0$, $x \geqq 0$, satisfies also $\alpha' x \leqq b_{m+1} + M$ automatically.

This means that for sufficiently large M, $\tilde{x}$, a feasible solution of (6.13), is also a feasible solution of the new problem. But, the objective function $c'\tilde{x}$ increases infinitly when M is increased, which contradicts with the extence of the optimal solution of the new problem. Q.E.D.

Secondly, we must treat the case where $\sum_{i=1}^{m+1} c_i b_{i,m+1} = 0$. In this case, if there is at least one negative among $b_{i,m+1}$ $(i=1, 2, \cdots, m+1)$, the above procedure can be applied.

On the contrary, if there is not any negative one among $\{b_{i,m+1}\}$, we apply the usual simplex procedure to replace any one vector among the optimal basis $\{\tilde{P}_i;\ i=1,\ 2,\ \cdots,\ m+1\}$ by $\tilde{\lambda}_{m+1}$. It is easily shown that this replacement does not destroy the feasibility and the optimality. The procedure after this replacement is quite the same as in Case (i).

7. Summary and Acknowledgements.

As stated in previous sections, various linear programing problems which are obtained from the original one by changing various coefficients may be solved comparably rapidly and easily by making use of the optimal solution of the original problem. But, if it takes too many stages of renewing tableau until we get the optimal tableau, we must take into account the accumulation of error of computation.

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