MODAL INTERVALS FOR CHI-SQUARE DISTRIBUTIONS

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1. Summary and introduction

Let $f(x)$, $-\infty<x<\infty$, be a probability density function which is continuous and unimodal. If numbers L and U ($L<U$) satisfy the conditions

$$\int_L^U f(x)\,dx=1-\alpha\ , \tag{1}$$

$$f(U)-f(L)=0\ , \tag{2}$$

we call $[L, U]$ the $(1-\alpha)$-content modal interval for $f(x)$. The following Table 3 lists the values of L and U, together which other values, for the χ^2 distribution with ϕ degrees of freedom

$$f_\phi(x)=\frac{1}{2\Gamma\left(\frac{\phi}{2}\right)}\left(\frac{x}{2}\right)^{\frac{\phi}{2}-1}e^{-\frac{x}{2}}\ , \qquad 0<x<\infty \tag{3}$$

for $1-\alpha=0.80$, 0.90, 0.95, 0.99 and $\phi=3(1)30(10)100$. We discuss in Section 2 some uses of the table, and in Sections 3 and 4 the method of its computation.

The term "modal interval" was used by Blackwell-Girshick [1], in connection with the problem of estimates of bounded error, for an a posteriori distribution without such restrictions as the existence of the probability density function etc. This notion is the same as that named "minimum range" by Geary [2]. The same kind of table as Table 3 was published by Girshick and others in [3]; however, it gives $1-\alpha$ for various values of $(U-L)/(U+L)$ and for $\phi=4(4)160$, rather than giving L and U for various values of $1-\alpha$. Diagrams of modal intervals for F-distributions were also published by Reiter [4]. Table 3 of this paper is intended to be supplemental to them.

2. Uses of modal intervals for χ^2 distributions

(i) Test on the scale parameter in gamma type distributions.

Let $X_1, X_2, \cdots, X_n$ be a sample from the population which has the p.d.f.

$$(4)\qquad \frac{1}{\theta}f_\phi\left(\frac{x}{\theta}\right)=\frac{1}{2\theta\Gamma\left(\frac{\phi}{2}\right)}\left(\frac{x}{2\theta}\right)^{\frac{\phi}{2}-1}e^{-\frac{x}{2\theta}}\ ,\qquad 0<x<\infty\ ,$$

$$0<\phi<\infty\ ,\quad 0<\theta<\infty\ .$$

If we restrict our statistics to $\sum_{i=1}^{n} X_i$, then, since $\sum_{i=1}^{n} X_i/\theta$ has the χ^2 distribution, the critical region of the unbiased most powerful level α test for the problem:

$$(5)\qquad \begin{array}{ll} \text{Hypothesis:} & \theta=\theta_0\ , \\ \text{Alternative:} & \theta\neq\theta_0\ , \end{array}$$

is shown (see e.g. [5] Chap. 5) to be

$$(6)\qquad R=\left\{(X_1, X_2, \cdots, X_n)\,\middle|\, \sum_{i=1}^{n} X_i\leqq L\theta_0 \text{ or } \sum_{i=1}^{n} X_i\geqq U\theta_0\right\}\ .$$

The values of L and U are determined so that the conditions

$$(7)\qquad \beta(\theta_0)=\alpha\ ,$$

$$(8)\qquad \left.\frac{\partial\beta}{\partial\theta}\right]_{\theta=\theta_0}=0\ ,$$

are satisfied, where $\beta(\theta)=P_\theta\{(X_1, X_2, \cdots, X_n)\in R\}$ denotes the power function of the test. Equations (7) and (8) can be expressed as

$$(9)\qquad \int_L^U f_{n\phi}(x)\,dx=1-\alpha\ ,$$

$$(10)\qquad Lf_{n\phi}(L)=Uf_{n\phi}(U)\ ,$$

or equivalently as

$$(11)\qquad \int_L^U f_{n\phi+2}(x)\,dx=1-\alpha\ .$$

$$(12)\qquad f_{n\phi+2}(L)=f_{n\phi+2}(U)\ .$$

(The equation (11) obtains when one integrates the equation (9) by parts and makes use of the equation (12).) Accordingly, $[L, U]$ should be the $(1-\alpha)$-content modal interval for χ^2 distribution with $(n\phi+2)$ degrees of freedom.

In particular, if $\phi=2$ (exponential distribution) the statistic $\sum_{i=1}^{n} X_i$ is sufficient and complete for the class $\{f(x/\theta)/\theta\ ;\ 0<\theta<\infty\}$, so we would not suffer from any loss in restricting our statistics only to $\sum_{i=1}^{n} X_i$.

As a related case, we consider the following problem. Let $X_1, X_2, \cdots, X_n$ be a sample from the normal population $N(\mu, \sigma^2)$, $-\infty<\mu<\infty$, $0<\sigma^2<\infty$. We test

$$\text{(13)}\qquad \begin{array}{lll} \text{Hypothesis}: & \sigma^2=\sigma_0^2, & -\infty<\mu<\infty, \\ \text{Alternative}: & \sigma^2\neq\sigma_0^2, & -\infty<\mu<\infty. \end{array}$$

For the class of normal distributions $N(\mu, \sigma^2)$, $-\infty<\mu<\infty$, $0<\sigma^2<\infty$, sum of squares $S=\sum_{i=1}^{n}(X_i-\bar{X})^2$ is sufficient (σ^2). Therefore, from Fraser's theorem concerning the generalized sufficient statistic (c.f. [6] Chap. 2.3), the test

$$\text{(14)}\qquad R=\{(X_1, X_2, \cdots, X_n)\,|\,S\leqq L\sigma_0^2 \text{ or } S\geqq U\sigma_0^2\},$$

where $[L, U]$ is the $(1-\alpha)$-content modal interval for χ^2 distribution *with* n+1 *degrees of freedom*, is the unbiased most powerful level α test for the problem (9).

Table 1. Power functions of the unbiased most powerful and the usual "$\alpha/2$ in both sides" test for the problem (5)*

$n\phi=4$

level	$\alpha=0.05$		$\alpha=0.20$	
θ_0/θ	Unbiased *MP*	"$\alpha/2-\alpha/2$"	Unbiased *MP*	"$\alpha/2-\alpha/2$"
0.1	.8652	.8923	.9260	.9427
0.2	.6356	.6949	.7788	.8220
0.3	.4319	.5047	.6239	.6861
0.4	.2819	.3520	.4891	.5590
0.6	.1187	.1630	.3059	.3644
0.8	.0617	.0798	.2213	.2516
1.0	.0500	.0500	.2000	.2000
1.2	.0562	.0445	.2140	.1879
1.6	.0863	.0596	.2872	.2239
2.0	.1243	.0857	.3771	.2913
4.0	.3424	.2528	.7356	.6273
6.0	.5434	.4264	.9026	.8276
8.0	.6976	.5769	.9667	.9254
10.0	.8060	.6963	.9891	.9690
	$\frac{\theta_0}{\theta}=1.177$	.0444 (min)	$\frac{\theta_0}{\theta}=1.185$	0.188 (min)

$n\phi=12$

level	$\alpha=0.05$		$\alpha=0.20$	
θ_0/θ	Unbiased *MP*	"$\alpha/2-\alpha/2$"	Unbiased *MP*	"$\alpha/2-\alpha/2$"
0.2	.9611	.9682	.9851	.9882
0.3	.8332	.8576	.9235	.9369
0.4	.6331	.6744	.8020	:8306
0.5	.4262	.4736	.6438	.6849
0.6	.2607	.3030	.4852	.5308
0.8	.0872	.1062	.2642	.2938
1.0	.0500	.0500	.2000	.2000
1.1	.0557	.0490	.2120	.1979
1.2	.0703	.0577	.2428	.2165
1.6	.1787	.1456	·4497	.3947
2.0	.3316	.2807	.6604	.6020
2.5	.5341	.4719	.8414	.7975
3.0	.7057	.6461	.9352	.9093
4.0	.9066	.8721	.9918	.9862
	$\frac{\theta_0}{\theta}=1.057$	.0480 (min)	$\frac{\theta_0}{\theta}=1.058$	.1959 (min)

* For the problem (13) these correspond to the cases $n=5\,(\phi=4)$ and $n=13\,(\phi=12)$, and the argument θ_0/θ should be read as σ_0^2/σ^2.

In Table 1, power functions of the most powerful unbiased test and of the usual "$\alpha/2$ in both sides" test are compared.

(ii) Estimate of bounded relative error.

Let again $X_1, X_2, \cdots, X_n$ be a sample from the population which has the p.d.f. $f(x/\theta)/\theta$ in equation (4), and $[L, U]$ be the $(1-\alpha)$-content modal interval for $f_{n\phi}(x)$. We consider the estimate of bounded relative error $\hat{\theta}$ of θ, that is the estimate which is optimal for the loss function

$$L(\hat{\theta}, \theta)=\begin{cases} 0 & \left|\dfrac{\hat{\theta}}{\theta}-1\right|=\gamma, \\ 1 & \text{otherwise}, \end{cases} \tag{15}$$

where

$$\gamma=\frac{U-L}{U+L}. \tag{16}$$

(For the given value of γ we may adjust n so that $\gamma \leqq (U-L)/(U+L)$.) The invariant (multiplicative) and minimax solution is (c.f. [1], [3]) $\hat{\theta}=\sum_{i=1}^{n} X_i$ and expected loss is equal to $E_\theta[L(\hat{\theta}, \theta)]=\alpha$.

(iii) Confidence interval which is the shortest in the mean.

From the same sample as in (ii) we construct a confidence interval for $1/\theta$ in the form $\left[L\Big/\sum_{i=1}^{n} X,\ U\Big/\sum_{i=1}^{n} X\right]$, and determine L and U so that its coefficient of confidence is equal to $1-\alpha$, and that its expected length is the shortest. These requirements are expressed as

$$\int_L^U f_{n\phi}(x)\,dx=1-\alpha \tag{17}$$

$$E_\theta\left[\frac{U}{\sum X_i}-\frac{L}{\sum X_i}\right]=\frac{U-L}{\theta(n\phi-2)}=\min \tag{18}$$

Therefore, $[L, U]$ should be the $(1-\alpha)$-content modal interval for $f_{n\phi}(x)$. The shortest confidence interval for θ in the form $\left[l\sum_{i=1}^{n} X_i,\ u\sum_{i=1}^{n} X_i\right]$ cannot be expressed through the modal interval as above.

In table 2 the length of each modal interval is compared with that of the corresponding interval $[L', U']$ such that

$$\int_0^{L'} f_\phi(x)\,dx=\int_{U'}^{\infty} f_\phi(x)\,dx=\frac{\alpha}{2}. \tag{19}$$

Table 2.

The ratio of the length of $[L', U']$ defined by the equation (19) to that of the modal interval $[L, U]$, for various content $1-\alpha$ and degrees of freedom ϕ.

ϕ \ $1-\alpha$	0.99	0.95	0.90	0.80
3	1.1250	1.1751	1.1956	1.2248
6	1.0742	1.0834	1.0874	1.0913
10	1.0453	1.0480	1.0492	1.0503
20	1.0226	1.0231	1.0234	1.0236
40	1.0111	1.0113	1.0114	1.0114
60	1.0075	1.0075	1.0075	1.0075
100	1.0045	1.0045	1.0045	1.0045

3. Method of computation

For the computation of modal intervals with given content, it seems appropriate to apply Cornish-Fisher expansion [7] for the "symmetrized" distribution, in the following way.

The limits of modal interval $[L. U]$ for the continuous unimodal p.d.f. $f(x)$, are one-valued functions of the length $2s=U-L$ of the interval. That is,

$$\begin{aligned} &U=U(s), \qquad L=L(s);\\ &U(s)-L(s)=2s,\\ &f(U(s))-f(L(s))=0. \end{aligned} \tag{20}$$

The "symmetrized" distribution $g(s)=G'(s)$ for $f(x)=F'(x)$ are defined from the equality

$$\begin{aligned} 2G(s)-1&=G(s)-G(-s)\\ &=F(U(s))-F(L(s)), \qquad 0\leqq s, \end{aligned} \tag{21}$$

or equivalently from

$$\begin{aligned} g(s)&=g(-s)\\ &=f(L(s))=f(U(s)), \qquad 0\leqq s. \end{aligned} \tag{22}$$

If we get the $100(1-\alpha/2)$ percentile $s_{\alpha/2}$ for $g(s)$, the $(1-\alpha)$-content modal interval $[L_\alpha, U_\alpha]$ for $f(x)$ are given as

$$L_\alpha=L(s_{\alpha/2}), \qquad U_\alpha=U(s_{\alpha/2}).$$

In our case, putting $f(x)$ in (22) equal to $f_\phi(x)$, we have

$$L(s)=\frac{2s}{e^{\frac{2s}{\phi-2}}-1}, \qquad U(s)=\frac{2se^{\frac{2s}{\phi-2}}}{e^{\frac{2s}{\phi-2}}-1} \tag{23}$$

and the corresponding symmetrized distribution

$$g_\phi(s)=\frac{1}{2\Gamma\left(\frac{\phi}{2}\right)}\left\{\frac{s}{e^{\frac{2s}{\phi-2}}-1}\right\}^{\frac{\phi}{2}-1}\exp\left\{\frac{-s}{e^{\frac{2s}{\phi-2}}-1}\right\}. \tag{24}$$

For simplicity we put

$$y=\frac{2s}{\phi-2}$$

and

$$\frac{y}{e^y-1}=1-\frac{y}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{(2n)!}y^{2n}=1+t\,.$$

Then an asymptotic expansion of $g(s)$ gives

$$\begin{aligned} g_\phi(s)&=\frac{\left(\frac{\phi-2}{2}\right)^{\frac{\phi}{2}-1}}{2\Gamma\left(\frac{\phi}{2}\right)}(1+t)^{\frac{\phi}{2}-1}\exp\left\{-\frac{\phi-2}{2}(1+t)\right\} \\ &=\frac{\left(\frac{\phi-2}{2}\right)^{\frac{\phi}{2}-2}e^{-\frac{\phi-2}{2}}}{2\Gamma\left(\frac{\phi}{2}\right)}\exp\left\{\frac{\phi-2}{2}\left(-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right)\right\} \\ &=K_\phi h(y)\,e^{-\frac{(\phi-2)y^2}{16}}\,. \end{aligned} \tag{25}$$

In the last expression, the first factor is equal to

$$K_\phi=\frac{\left(\frac{\phi-2}{2}\right)^{\frac{\phi-2}{2}}e^{-\frac{\phi-1}{2}}}{2\Gamma\left(\frac{\phi}{2}\right)}=\frac{1}{\sqrt{4\pi(\phi-2)}\,S\left(\frac{\phi-2}{2}\right)} \tag{26}$$

where $S(x)$ is the Stirling series

$$S(x)=1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\cdots\,.$$

The second factor of the righthand member of (25) is equal to

$$\begin{aligned} h_\phi(y)&=\exp\left\{\frac{\phi-2}{2}\left(\frac{y^2}{8}-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\right)\right\} \\ &=\exp\left\{\frac{\phi-2}{2}\left(\frac{y^4}{2^6\cdot 3^2}-\frac{y^6}{2^6\cdot 3^4\cdot 5}+\frac{y^8}{2^{11}\cdot 3\cdot 5^2\cdot 7}\right.\right. \\ &\qquad\left.\left.-\frac{y^{10}}{2^{10}\cdot 3^5\cdot 5^2\cdot 7}+\frac{691y^{12}}{2^{13}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 11}+\cdots\right)\right\}. \end{aligned} \tag{27}$$

These results show that a random variable $\sqrt{(\phi-2)/8}\,y=s/\sqrt{2(\phi-2)}$ has the asymptotic distribution $N(0,1)$. From (25) we have the moments of $g_\phi(s)$, as

$$\mu_{2k}=\int_{-\infty}^{\infty} s^{2k} g_\phi(s)\,ds = \frac{2^k(2k-1)!!\,(\phi-2)^k}{S\left(\frac{\phi-2}{2}\right)} M_{2k}\,. \tag{28}$$

where

$$\begin{aligned}
M_{2k}=1&+\frac{(2k+3)(2k+1)}{2\cdot 3^2\cdot(\phi-2)}-\left\{\frac{2^2(2k+5)(2k+3)(2k+1)}{3^4\cdot 5}\right.\\
&\left.-\frac{(2k+7)(2k+5)\cdots(2k+1)}{2^3\cdot 3^4}\right\}\frac{1}{(\phi-2)^2}\\
&+\left\{\frac{(2k+7)(2k+5)\cdots(2k+1)}{3\cdot 5^2\cdot 7}-\frac{2(2k+9)(2k+7)\cdots(2k+1)}{3^6\cdot 5}\right.\\
&\left.+\frac{(2k+11)(2k+9)\cdots(2k+1)}{2^4\cdot 3^7}\right\}\frac{1}{(\phi-2)^3}\\
&-\left\{\frac{2^4(2k+9)(2k+7)\cdots(2k+1)}{3^5\cdot 5^2\cdot 7}-\frac{(2k+11)(2k+9)\cdots(2k+1)}{2\cdot 3^3\cdot 5^2\cdot 7}\right.\\
&+\frac{2^3(2k+11)(2k+9)\cdots(2k+1)}{3^8\cdot 5^2}+\frac{(2k+13)(2k+11)\cdots(2k+1)}{2\cdot 3^8\cdot 5}\\
&\left.-\frac{(2k+15)(2k+13)\cdots(2k+1)}{2^7\cdot 3^9}\right\}\frac{1}{(\phi-2)^4}\\
&+\left\{\frac{2^4\cdot 691(2k+11)(2k+9)\cdots(2k+1)}{3^7\cdot 5^3\cdot 7^2\cdot 11}-\frac{2^3(2k+13)(2k+11)\cdots(2k+1)}{3^7\cdot 5^2\cdot 7}\right.\\
&-\frac{2^2(2k+13)(2k+11)\cdots(2k+1)}{3^5\cdot 5^3\cdot 7}+\frac{(2k+15)(2k+13)\cdots(2k+1)}{2^3\cdot 3^5\cdot 5^2\cdot 7}\\
&+\frac{2^2(2k+15)(2k+13)\cdots(2k+1)}{3^{10}\cdot 5^2}-\frac{(2k+17)(2k+15)\cdots(2k+1)}{2^2\cdot 3^{11}\cdot 5}\\
&\left.+\frac{(2k+19)(2k+17)\cdots(2k+1)}{2^8\cdot 3^{11}\cdot 5}\right\}\frac{1}{(\phi-2)^5}+\cdots
\end{aligned} \tag{29}$$

For Cornish-Fisher expansion, it is necessary to compute the values of

$$\begin{aligned}
d&=\frac{\kappa_4}{\mu_2^2}=3\left[S\frac{M_4}{M_2^2}-1\right]=\frac{4}{3(\phi-2)}+\frac{8}{9(\phi-2)^2}+\cdots\\
f&=\frac{\kappa_6}{\mu_2^3}=15\left[S^2\frac{M_6}{M_2^3}-3S\frac{M_4}{M_2^2}+2\right]=\frac{32}{3(\phi-2)^2}+\frac{704}{45(\phi-2)^3}+\cdots
\end{aligned} \tag{30}$$

etc., where κ_i is the i-th cumulant and $S=S\left(\frac{\phi-2}{2}\right)$. The expansion

Table 3. $(1-\alpha)$-Content Modal Intervals for χ^2 Distributions with d.f. ϕ

ϕ	$1-\alpha=0.99$				$1-\alpha=0.95$			
	L	U	$P_r\{x^2 \leq L\}$	$\frac{U-L}{U+L}$	L	U	$P_r\{x^2 \leq L\}$	$\frac{U-L}{U+L}$
3	0.0001	11.348	0.00000	1.0000	0.0316	7.818	0.00005	0.9992
4	0.0175	13.285	0.00004	0.9974	0.0847	9.530	0.00087	0.9824
5	0.1011	15.126	0.00016	0.9867	0.2960	11.195	0.00228	0.9485
6	0.2640	16.901	0.00035	0.9692	0.6070	12.802	0.00372	0.9095
7	0.4955	18.630	0.00053	0.9482	0.9891	14.369	0.00498	0.8712
8	0.7856	20.296	0.00073	0.9255	1.4250	15.897	0.00612	0.8355
9	1.1220	21.931	0.00090	0.9027	1.9026	17.392	0.00708	0.8028
10	1.4978	23.533	0.00106	0.8803	2.4138	18.861	0.00793	0.7731
11	1.9068	25.106	0.00120	0.8588	2.9532	20.305	0.00867	0.7461
12	2.3444	26.653	0.00133	0.8383	3.5162	21.729	0.00933	0.7214
13	2.8069	28.178	0.00145	0.8188	4.0994	23.135	0.00991	0.6989
14	3.2912	29.683	0.00156	0.8004	4.7005	24.525	0.01044	0.6783
15	3.7949	31.170	0.00166	0.7829	5.3171	25.900	0.01092	0.6593
16	4.3161	32.641	0.00176	0.7662	5.9477	27.263	0.01135	0.6418
17	4.8530	34.097	0.00184	0.7508	6.5908	28.614	0.01175	0.6256
18	5.4041	35.540	0.00192	0.7360	7.2453	29.955	0.01211	0.6105
19	5.9683	36.971	0.00200	0.7220	7.9100	31.285	0.01245	0.5964
20	6.5447	38.388	0.00206	0.7087	8.5842	32.607	0.01276	0.5832
21	7.1315	39.798	0.00213	0.6961	9.2670	33.921	0.01305	0.5709
22	7.7293	41.196	0.00219	0.6840	9.9579	35.227	0.01332	0.5592
23	8.3358	42.586	0.00225	0.6726	10.656	36.525	0.01357	0.5483
24	8.9515	43.967	0.00230	0.6617	11.361	37.818	0.01381	0.5380
25	9.5770	45.336	0.00236	0.6512	12.073	39.103	0.01403	0.5282
26	10.207	46.706	0.00240	0.6413	12.791	40.383	0.01424	0.5189
27	10.846	48.064	0.00245	0.6318	13.514	41.658	0.01444	0.5101
28	11.492	49.416	0.00249	0.6226	14.243	42.927	0.01463	0.5017
29	12.145	50.761	0.00253	0.6139	14.977	44.192	0.01481	0.4938
30	12.803	52.100	0.00257	0.6055	15.716	45.451	0.01498	0.4861
40	19.670	65.219	0.00288	0.5366	23.319	57.836	0.01631	0.4253
50	26.919	77.961	0.00310	0.4867	31.218	69.931	0.01722	0.3827
60	34.437	90.437	0.00326	0.4484	39.323	81.820	0.01790	0.3508
70	42.159	102.71	0.00339	0.4180	47.585	93.555	0.01842	0.3257
80	50.040	114.83	0.00349	0.3930	55.970	105.17	0.01885	0.3053
90	58.052	126.81	0.00357	0.3720	64.454	116.68	0.01920	0.2883
100	66.172	138.69	0.00365	0.3540	73.021	128.11	0.01949	0.2740

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ϕ	$1-\alpha=0.90$				$1-\alpha=0.80$			
	L	U	$P_r\{x^2\leq L\}$	$\frac{U-L}{U+L}$	L	U	$P_r\{x^2\leq L\}$	$\frac{U-L}{U+L}$
3	0.0121	6.262	0.00035	0.9961	0.0457	4.673	0.00256	0.9806
4	0.1676	7.864	0.00332	0.9583	0.3346	6.161	0.01253	0.8970
5	0.4766	9.432	0.00698	0.9038	0.7790	7.621	0.02160	0.8145
6	0.8827	10.958	0.01033	0.8509	1.3078	9.042	0.02877	0.7473
7	1.3560	12.436	0.01290	0.8034	1.8910	10.428	0.03436	0.6930
8	1.8746	18.892	0.01538	0.7622	2.5134	11.784	0.03890	0.6484
9	2.4313	15.314	0.01732	0.7260	3.1652	13.117	0.04262	0.6112
10	3.0173	16.711	0.01899	0.6941	3.8405	14.430	0.04575	0.5796
11	3.6276	18.087	0.02043	0.6659	4.5351	15.727	0.04842	0.5524
12	4.2582	19.446	0.02169	0.6407	5.2458	17.009	0.05074	0.5286
13	4.9063	20.789	0.02280	0.6181	5.9703	18.279	0.05277	0.5076
14	5.5696	22.119	0.02380	0.5977	6.7070	19.537	0.05458	0.4889
15	6.2462	23.436	0.02469	0.5791	7.4539	20.786	0.05619	0.4721
16	6.9347	24.742	0.02550	0.5622	8.2103	22.026	0.05764	0.4569
17	7.6339	26.039	0.02624	0.5466	8.9751	23.258	0.05896	0.4431
18	8.3427	27.326	0.02691	0.5322	9.7473	24.483	0.06016	0.4305
19	9.0603	28.605	0.02754	0.5189	10.527	25.701	0.06127	0.4189
20	9.7859	29.876	0.02811	0.5065	11.312	26.914	0.06229	0.4081
21	10.519	31.140	0.02864	0.4950	12.104	28.120	0.06323	0.3982
22	11.259	32.398	0.02914	0.4842	12.900	29.322	0.06410	0.3889
23	12.005	33.649	0.02960	0.4741	13.702	30.519	0.06492	0.3803
24	12.756	34.895	0.03003	0.4646	14.508	31.711	0.06568	0.3722
25	13.514	36.136	0.03044	0.4556	15.319	32.899	0.06639	0.3646
26	14.276	37.372	0.03082	0.4472	16.133	34.083	0.06707	0.3574
27	15.044	38.603	0.03119	0.4392	16.952	35.264	0.06770	0.3507
28	15.816	39.830	0.03153	0.4316	17.774	36.441	0.06829	0.3443
29	16.592	41.052	0.03185	0.4243	18.599	37.615	0.06886	0.3383
30	17.372	42.271	0.03216	0.4175	19.427	38.786	0.06940	0.3325
40	25.357	54.276	0.03457	0.3632	27.855	50.352	0.07358	0.2877
50	33.591	66.037	0.03620	0.3257	36.477	61.726	0.07642	0.2571
60	41.999	77.625	0.03741	0.2978	45.236	72.965	0.07850	0.2346
70	50.539	89.083	0.03835	0.2761	54.098	84.102	0.08011	0.2171
80	59.182	100.44	0.03911	0.2585	63.039	95.159	0.08141	0.2030
90	67.909	111.71	0.03973	0.2439	72.047	106.150	0.08248	0.1914
100	76.706	122.91	0.04026	0.2315	81.109	117.09	0.08339	0.1815

formula, including the term of the order of d only or including further the terms of the order of f and d^2, are sufficient to get enough accurate values of $s_{\alpha/2}$ for larger ϕ. To included more terms in expansion does not seem to improve the approximation for ϕ smaller than 100. For smaller ϕ iteration processes making use of Pearson's or Sluckiî's Tables of the Incomplete Γ-Function [8, 9] were necessary. Most of the computation works was accomplished with the relay computer FACOM-128 in our Institute.

4. Computation for the distribution with smaller d. f.

For the distribution with d. f. less than 10, we had better apply the interation methods to get $s_{\alpha/2}$. However, as the initial value of the iteration we used also Cornish-Fisher expansion, seeking another expansion formula for the moment.

For simplicity, let

$$\nu=\frac{\phi}{2}-1\,,\quad y=\frac{2s}{\phi-2}=\frac{s}{\nu}\,,$$

then

$$(31)\qquad g_\phi(s)=\frac{\nu^\nu}{2\Gamma(\nu+1)}\left(\frac{y}{e^y-1}\right)^\nu \exp\left(-\frac{\nu y}{e^y-1}\right)$$
$$=\frac{\nu^\nu}{2\Gamma(\nu+1)}\sum_{i=0}^{\infty}\frac{(-\nu)^i}{i!}\sum_{j=0}^{\infty}\binom{\nu+i+j-1}{j}y^{\nu+i}e^{-(\nu+i+j)y}$$

Let λ be an even integer. Then, the moment of s is expressed as

$$(32)\qquad \mu_\lambda=\int_{-\infty}^{\infty}s^\lambda g_\phi(s)ds\,.$$
$$=\frac{\nu^{\nu+\lambda+1}}{\Gamma(\nu+1)}\sum_{i=0}^{\infty}\frac{(-\nu)^i}{i!}\sum_{j=0}^{\infty}(\lambda+\nu+i)^{(\lambda+1)}\frac{\Gamma(\nu+i+j)}{j!(\nu+i+j)^{\lambda+\nu+i+1}}\,,$$

where $x^{(n)}$ is the factorial function

$$x^{(n)}=x(x-1)\cdots(x-n+1)\,.$$

Putting $l=i+j$ we rearrange the above equation to have

$$\mu_\lambda=\frac{\Gamma(\nu+\lambda+1)}{(\nu+1)}+\frac{\nu^{\alpha+\nu+1}}{\Gamma(\nu+1)}\sum_{l=1}^{\infty}\frac{\Gamma(\nu+l)}{l!(\nu+l)^{\nu+\lambda}}\sum_{i=0}^{\infty}(\lambda+\nu+i)^{(\lambda+1)}\left(\frac{-\alpha}{\alpha+l}\right)^i\,.$$

Applying to the last summation the formulas

$$(x+y+n-1)^{(n)}=\sum_{h=0}^{n}\binom{n}{h}(x+n-1)^{(n-h)}y^{(h)}$$

and

$$\sum_{i=0}^{l}\binom{l}{i}i^{(h)}x^{i}(x+y)^{l-i}=l^{(h)}x^{h}(x+y)^{l-h},$$

we get

$$\mu_\lambda=\frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}+\frac{\nu^{\nu+\lambda+1}}{\Gamma(\nu+1)}\sum_{l=1}^{\infty}\frac{l^{l}\Gamma(\nu+l)}{l!(\nu+l)^{l+\nu+\lambda+1}} \times\sum_{h=0}^{\lambda+1}\binom{\lambda+1}{h}(\nu+\lambda)^{(\lambda+1-h)}l^{(h)}\left(\frac{-\nu}{l}\right)^{h} \tag{33}$$

If we introduce the numbers ϕ_j^h defined by the relation

$$z^{(h)}=\sum_{j=1}^{h}\phi_j^h z^j$$

or

$$\begin{cases}\phi_h^h=1\\ \phi_j^{h+1}=\phi_{j-1}^h-h\phi_j^h\\ \phi_1^{h+1}=-h\phi_0^h=(-1)^h h!\end{cases}$$

then we finally get the expression suitable for computation

$$\begin{aligned}\mu_\lambda=&\frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}+\frac{\nu^{\nu+\lambda+1}}{\Gamma(\nu+1)}\\ &\times\Bigg[\sum_{h=0}^{\lambda+1}\tbinom{\lambda+1}{h}(\nu+\lambda)^{(\lambda+1-h)}(-\nu)^{h}\sum_{l=1}^{\infty}\frac{l^{l}\Gamma(\nu+l)}{l!(\nu+l)^{l+\nu+\lambda+1}}\\ &+\sum_{j=1}^{\lambda}\left\{(-\nu)^{j+1}\sum_{h=j+1}^{\lambda+1}\binom{\lambda+1}{h}\phi_{h-j}^{h}(\nu+\lambda)^{(\lambda+1-h)}(-\nu)^{h-j-1}\right\}\\ &\times\sum_{l=1}^{\infty}\frac{l^{l-j}\Gamma(\nu+l)}{l!(\nu+l)^{l+\nu+\lambda+1}}\Bigg].\end{aligned} \tag{34}$$

As special cases

$$\begin{aligned}\mu_2=&(\nu+2)(\nu+1)-\nu^{\nu+4}[4\gamma(0,2,\nu)+6\nu\gamma(1,2,\nu)+2\nu^2\gamma(2,2,\nu)]\\ \mu_4=&(\nu+4)(\nu+3)(\nu+2)(\nu+1)\\ &+\nu^{\nu+6}[8(5\nu-12)\gamma(0,4,\nu)+20\nu(5\nu-12)\gamma(1,4,\gamma)\\ &+80\nu^2(\nu-3)\gamma(2,4,\nu)+20\nu^3(\nu-6)\gamma(3,4,\nu)-24\nu^4\gamma(4,4,\nu)]\end{aligned} \tag{35}$$

etc., where

$$\gamma(j, \lambda, \nu) = \sum_{l=1}^{\infty} \frac{l^{l-j}\Gamma(\nu+l)}{l!\Gamma(\nu+1)(\nu+l)^{l+\nu+\lambda+1}} \tag{36}$$

$$\nu = \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

$$\lambda = 2, 4, 6, \cdots$$

$$j = 0, 1, \cdots, \lambda .$$

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THE INSTITUTE OF STATISTICAL MATHEMATICS

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ERRATA

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P. 117–125, throughout the paper, read "$o\left(\frac{1}{n}\right)$" instead of "$0\left(\frac{1}{n}\right)$"