NOTE ON THE UTILIZATION OF THE GENERALIZED STUDENT RATIO IN THE ANALYSIS OF VARIANCE OR DISPERSION

By MINORU SIOTANI

(Received Dec. 20, 1957)

Summary. In this note, we shall consider the usage of the generalized Student ratio T or T^2, which is the statistic introduced by H. Hotelling [3, 4], in the analysis of variance or dispersion. F. A. Graybill and J. L. Folks [1, 2] have shown that in a randomized block experiment and in the situation of heterogeneneity of error variances, the equality of main effect constants can be tested by using the T^2-statistic. It will be shown that their approach is easily extended for testing the significance of the interaction in a two-factor experiment with different error variances. For the multivariate analysis of dispersion, the use of the generalized T^2 measure is illustrated by a simple example and the asymptotic power of the test based on this statistic is also considered. The tables of 5% and 1% significance points for this test are prepared for three and four dimensional cases.

1. Two-factor experiment.

Let us consider that we have a factorial experiment with two factors A and B, A being at p levels and B at q levels and moreover the pattern composed of those pq treatments be carried out, for example, at each of n locations or in each of n days. We assume that the individual $x_{ij\nu}(i=1, 2, \cdots, p; j=1, 2, \cdots, q; \nu=1, 2, \cdots, n)$ have the mathematical model

$$x_{ij\alpha}=\mu+\alpha_i+\beta_j+\gamma_{ij}+\delta_\nu+\varepsilon_{ij\nu} \tag{1}$$

with ordinary restrictions $\sum_{i=1}^{p}\alpha_i=\sum_{j=1}^{q}\beta_j=\sum_{i=1}^{p}\gamma_{ij}=\sum_{j=1}^{q}\gamma_{ij}=0$, where μ is the general mean, α_i is the effect of the i th A, β_j is the effect of the j th B, γ_{ij} is the interaction between the i th A and the j th B, δ_ν is the effect of the ν th location or the ν th day (assumed random) and $\varepsilon_{ij\nu}$ $(\nu=1, 2, \cdots, n)$ are error terms which are assumed to be distributed according to a pq variate normal distribution with mean $\mathbf{0}=(0, 0, \cdots, 0)$ and with covariance matrix Λ_1. Using the ordinary rule of notation, we shall denote the mean of $x_{ij\nu}$ with respect to some subscripts by x with the subscripts over which the mean operation is made replaced by dots.

Then we can test the hypothesis H_1 that all the α_i are equal, that is, $\alpha_1=\alpha_2 \cdots =\alpha_p(=0)$ in the following way: defining new variables such that

$$y_{i\cdot\nu}=x_{i\cdot\nu}-x_{p\cdot\nu}=\alpha_i-\alpha_p+\varepsilon_{i\cdot\nu}-\varepsilon_{p\cdot\nu} \quad (i=1, 2, \cdots, p-1; \nu=1, 2, \cdots n), \tag{2}$$

the row vector $\boldsymbol{y}_\nu=(y_{1\cdot\nu}, y_{2\cdot\nu}, \cdots, y_{(p-1)\cdot\nu})$ can be considered a random sample of size n from a $(p-1)$-variate normal population with mean $(\alpha_1-\alpha_p, \alpha_2-\alpha_p, \cdots, \alpha_{p-1}-\alpha_p)$ and with covariance matrix Λ_2. Then we can test H_1 by the Hotelling's T^2-statistic

$$T^2=(n-1)\bar{\boldsymbol{y}}\, S^{-1}\, \bar{\boldsymbol{y}}' \tag{3}$$

where $\bar{\boldsymbol{y}}=(y_{1\cdot\cdot}, y_{2\cdot\cdot}, \cdots, y_{(p-1)\cdot\cdot})$, $S=\frac{1}{n}\sum_{\nu=1}^{n}(\boldsymbol{y}_\nu-\bar{\boldsymbol{y}})'(\boldsymbol{y}_\nu-\bar{\boldsymbol{y}})$, S^{-1} is the inverse matrix of S and $\bar{\boldsymbol{y}}'$ is the transpose of $\bar{\boldsymbol{y}}$, and hence we can also test H_1 by the criterion

$$F=\frac{(n-p+1)}{p-1}\frac{T^2}{n-1}=\frac{(n-p+1)n}{p-1}\bar{\boldsymbol{y}}\left[\sum_{\nu=1}^{n}(\boldsymbol{y}_\nu-\bar{\boldsymbol{y}})'(\boldsymbol{y}_\nu-\bar{\boldsymbol{y}})\right]^{-1}\bar{\boldsymbol{y}}' \tag{4}$$

which has Snedecor's F distribution with $p-1$ and $n-p+1$ degrees of freedom under H_1 if $n>p-1$.

If we want to test the hypothesis H_2 that $\beta_1=\beta_2=\cdots=\beta_q(=0)$, we can do it in the same way as above.

Next we shall consider the test of the hypothesis H_3 that all the γ_{ij} are zero, the α_i and the β_j remaining unspecified. To do this we consider the $(p-1)(q-1)$ new variables such that

$$\begin{aligned} y_{ij\nu}&=x_{ij\nu}-x_{iq\nu}-x_{pj\nu}+x_{pq\nu} \\ &=\gamma_{ij}-\gamma_{iq}-\gamma_{pj}+\gamma_{pq}+(\varepsilon_{ij\nu}-\varepsilon_{iq\nu}-\varepsilon_{pj\nu}+\varepsilon_{pq\nu}) \\ & i=1, 2, \cdots, p-1;\ j=1, 2, \cdots, q-1;\ \nu=1, 2, \cdots, n. \end{aligned} \tag{5}$$

It is easily verified that the original hypothesis H_3 is equivalent to the hypothesis H_3' that $\phi_{ij}\equiv\gamma_{ij}-\gamma_{iq}-\gamma_{pj}+\gamma_{pq}$ are all zero as follows: if all the γ_{ij} are zero, it is obvious that all the ϕ_{ij} are zero. If $\phi_{ij}=0$ for $i=1, 2, \cdots, p-1$ and $j=1, 2, \cdots, q-1$, we have

$$\sum_{j=1}^{q-1}\phi_{ij}=\sum_{j=1}^{q-1}\gamma_{ij}-(q-1)\gamma_{iq}-\sum_{j=1}^{q-1}\gamma_{pj}+(q-1)\gamma_{pq}=q(\gamma_{pq}-\gamma_{iq})=0,$$

since $\sum_{j=1}^{q}\gamma_{ij}=0$. Therefore $\gamma_{iq}=\gamma_{pq}$ and $\gamma_{ij}=\gamma_{pj}$. Similarly, from $\sum_{i=1}^{p-1}\phi_{ij}=0$

and $\phi_{ij}=0$, we have $\gamma_{pj}=\gamma_{pq}$ and $\gamma_{ij}=\gamma_{iq}$. Hence $\gamma_{ij}=\gamma_{iq}=\gamma_{pj}=\gamma_{pq}$, that is, all the γ_{ij} $(i=1, 2, \cdots, p; j=1, 2, \cdots, q)$ are zero.

Since we can consider the set of $y_{ij\nu}$ for fixed ν as the $(p-1)(q-1)$ -dimensional normal variate with mean ϕ_{ij} and covariance matrix Λ_3, we can test H_3' by using the Hotelling's T^2

$$T^2=(n-1)\overline{\boldsymbol{Y}}\,\boldsymbol{V}^{-1}\,\overline{\boldsymbol{Y}}' \tag{6}$$

and hence by

$$\begin{aligned} F&=\frac{n-(p-1)(q-1)}{(p-1)(q-1)}\frac{T^2}{n-1} \\ &=\frac{[n-(p-1)(q-1)]n}{(p-1)(q-1)}\overline{\boldsymbol{Y}}\Big[\sum_{\nu=1}^{n}(\boldsymbol{Y}_\nu-\overline{\boldsymbol{Y}})'(\boldsymbol{Y}_\nu-\overline{\boldsymbol{Y}})\Big]^{-1}\overline{\boldsymbol{Y}}' \end{aligned} \tag{7}$$

which has F distribution with $(p-1)(q-1)$ and $\{n-(p-1)(q-1)\}$ degrees of freedom, where $\overline{\boldsymbol{Y}}$ is the mean of $\boldsymbol{Y}_\nu$ which is the row vector with components $y_{ij\nu}$ in some arrangement and $\boldsymbol{V}$ is the covariance matrix of $\boldsymbol{Y}_\nu$ $(\nu=1, 2, \cdots, n)$.

2. The multivariate analysis of dispersion.

Let us consider, for example, a randomized block design in which measurements are made in p dimensions and we assume that individual results $\boldsymbol{x}_{ij}=(x_{ij}^{(1)}, x_{ij}^{(2)}, \cdots, x_{ij}^{(p)})$ are given by the mathematical model

$$\boldsymbol{x}_{ij}=\boldsymbol{\mu}+\boldsymbol{\tau}_i+\boldsymbol{\beta}_j+\boldsymbol{\varepsilon}_{ij} \qquad (i=1, 2, \cdots, r; j=1, 2, \cdots, s) \tag{8}$$

with $\sum_{i=1}^{r}\boldsymbol{\tau}_i=\sum_{j=1}^{s}\boldsymbol{\beta}_i=\mathbf{0}$, wihere

$\boldsymbol{\mu}=(\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(p)})$ is the general mean, $\boldsymbol{\tau}_i=(\tau_i^{(1)}, \tau_i^{(2)}, \cdots, \tau_i^{(p)})$ is the effect of the i th treatment, $\boldsymbol{\beta}_j=(\beta_j^{(1)}, \beta_j^{(2)}, \cdots, \beta_j^{(p)})$ is the effect of the j th block, and $\boldsymbol{\varepsilon}_{ij}$'s are normally and independently distributed around $\mathbf{0}$ with a covariance matrix Λ. By the usual procedure we have

$$\begin{aligned} \sum_{i=1}^{r}\sum_{j=1}^{s}(\boldsymbol{x}_{ij}-\boldsymbol{x}_{..})'(\boldsymbol{x}_{ij}-\boldsymbol{x}_{..})&=s\sum_{i=1}^{r}(\boldsymbol{x}_{i.}-\boldsymbol{x}_{..})'(\boldsymbol{x}_{i.}-\boldsymbol{x}_{..})+r\sum_{j=1}^{s}(\boldsymbol{x}_{.j}-\boldsymbol{x}_{..})'(\boldsymbol{x}_{.j}-\boldsymbol{x}_{..}) \\ &\quad+\sum_{i=1}^{r}\sum_{j=1}^{s}(\boldsymbol{x}_{ij}-\boldsymbol{x}_{i.}-\boldsymbol{x}_{.j}+\boldsymbol{x}_{..})'(\boldsymbol{x}_{ij}-\boldsymbol{x}_{i.}-\boldsymbol{x}_{.j}+\boldsymbol{x}_{..}) \\ &=\boldsymbol{T}+\boldsymbol{B}+\boldsymbol{E}\,. \end{aligned} \tag{9}$$

$\boldsymbol{E}$ has the Wishart's distribution with $(r-1)(s-1)$ degrees of freedom and $\boldsymbol{E}/(r-1)(s-1)$ is the unbiased estimate of Λ. $\boldsymbol{T}$ and $\boldsymbol{B}$ have the Wishart's distributions with $(r-1)$ and $(s-1)$ degrees of freedom if $\boldsymbol{\tau}$'s and $\boldsymbol{\beta}$'s are zero, respectively. For testing the hypothesis that the effects of treatment are all zero, that is, $H_t: \boldsymbol{\alpha}_1=\boldsymbol{\alpha}_2=\cdots=\boldsymbol{\alpha}_r(=0)$, the Wilks' criterion $W=|\boldsymbol{E}|/|\boldsymbol{T}+\boldsymbol{E}|$, which is the maximum likelihood criterion, is used in the present practical application. But we can also test this hypothesis based on Hotelling's T^2 as follows: from (9) we have

$$\text{(10)} \qquad \operatorname{tr} \Lambda^{-1}\Big(\sum_{i=1}^{r}\sum_{j=1}^{s}(\boldsymbol{x}_{ij}-\boldsymbol{x}..)'(\boldsymbol{x}_{ij}-\boldsymbol{x}..)\Big)=\operatorname{tr}\Lambda^{-1}\boldsymbol{T}+\operatorname{tr}\Lambda^{-1}\boldsymbol{B}+\operatorname{tr}\Lambda^{-1}\boldsymbol{E}\,.$$

It is easily seen that $\operatorname{tr}\Lambda^{-1}\boldsymbol{T}$ has χ^2 distribution with $(r-1)p$ degrees of freedom when H_t is true, $\operatorname{tr}\Lambda^{-1}\boldsymbol{B}$ has χ^2-distribution with $(s-1)p$ degrees of freedom if all the β's are zero and $\operatorname{tr}\Lambda^{-1}\boldsymbol{E}$ is χ^2 with $(r-1)(s-1)p$ degrees of freedom regardless of whether the τ's and β's are zero or not. Since we have not a priori the knowledge of Λ, we must replace Λ by its unbiased estimate and in our present case we may use $\boldsymbol{L}=\dfrac{1}{(r-1)(s-1)}\boldsymbol{E}$ for it. Then we have the relation

$$\text{(11)} \qquad \operatorname{tr}\boldsymbol{L}^{-1}\Big(\sum_{i=1}^{r}\sum_{j=1}^{s}(\boldsymbol{x}_{ij}-\boldsymbol{x}..)'(\boldsymbol{x}_{ij}-\boldsymbol{x}..)\Big)=\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}+\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{B}+(r-1)(s-1)p\,.$$

Thus $\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}$ is used as a criterion for testing the hypothesis H_t and $\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{B}$ is used as a criterion for testing the hypothesis that $\boldsymbol{\beta}_1=\cdots=\boldsymbol{\beta}_s(=0)$. Now we consider the sampling distribution of $\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}$ under the null hypothesis H_t. It is easily seen that, by suitable orthogonal transformation, $\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}$ may be written as

$$\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}=\operatorname{tr}\Big(\frac{1}{a}\sum_{\nu=1}^{a}\boldsymbol{y}_\nu'\boldsymbol{y}_\nu\Big)^{-1}\Big(\sum_{\alpha=1}^{r-1}\boldsymbol{z}_\alpha'\boldsymbol{z}_\alpha\Big)=\sum_{\alpha=1}^{r-1}\boldsymbol{z}_\alpha\Big(\frac{1}{a}\sum_{\nu=1}^{a}\boldsymbol{y}_\nu'\boldsymbol{y}_\nu\Big)^{-1}\boldsymbol{z}_\alpha'$$

where $a=(r-1)(s-1)$ and $\boldsymbol{y}_\nu(\nu=1,2,\cdots,r-1)$ and $\boldsymbol{z}_\alpha(\alpha=1,2,\cdots,a)$ are independently and normally distributed with zero mean vector and with covariance matrix Λ. Therefore $\operatorname{tr}\boldsymbol{L}^{-1}\boldsymbol{T}$ may be considered as the sum of $(r-1)$ Hotelling's T^2 statistics, i.e., $T_1^2+T_2^2+\cdots+T_{r-1}^2$, components of which are depend on the same unbiased estimate of Λ. The criterions of this type were initially considered by Hotelling [4] and its exact sampiing distribution is known when $p=2$ and moreover, the tables of percentage points of the criterion for testing are now available in this case. For $p\geqq 3$, the approximate formula of percentage points of the criterion, which was obtained independently by the author [7, 8] and *K.* Ito [5], is prepared for use. If we denote, in general, the statistic considered above by

$$T_0^2 = m \operatorname{tr} \boldsymbol{L}^{-1}\boldsymbol{V} = \operatorname{tr} \boldsymbol{L}^{-1}\Big(\sum_{\nu=1}^{m} \boldsymbol{y}_\nu' \boldsymbol{y}_\nu\Big) , \tag{12}$$

where $\boldsymbol{L}$ and $\boldsymbol{V}$ are two independent unbiased estimates of the population covariance matrix Λ with n and m degrees of freedom, respectively. Then the approximate formula of the $\eta \times 100$ percentage points, $T_0^2(\eta)$, is given as

$$\begin{aligned} T_0^2(\eta) = x^2 &+ \frac{m}{2n}[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2)] \\ &+ \frac{m}{n^2}\Big\{\frac{m}{16}\Big(1-\frac{mp-2}{\chi^2}\Big)[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2]^2 \\ &- \frac{m}{8}[p(p+1)(\chi_4+\chi_2)+mp(\chi_4-\chi_2)][p+1)(x_4-1)+mp(\chi_4-2\chi_2+1)] \\ &- \frac{1}{3}[p(p^2+4)(\chi_6+\chi_4+\chi_2)+3mp(p+1)(\chi_6-\chi_2)+m^2p(\chi_6-2\chi_4+\chi_2)] \\ &+ \frac{1}{16}[4p(2p^2+5p+5)(\chi_8+\chi_6+\chi_4+\chi_2)+16mp(p+1)(\chi_8-\chi_2) \\ &+ mp(p^3+2p^2+5p+4)(\chi_8+\chi_6-\chi_4-\chi_2) \\ &+ 2m^2p(p^2+p+4)(\chi_8-\chi_6-\chi_4+\chi_2)+m^3p^2(\chi_8-3\chi_6+3\chi_4-\chi_2)]\Big\} \\ &+ O(n^{-3}) , \end{aligned} \tag{13}$$

where $\chi^2 \equiv \chi^2_{mp}(\eta)$, that is, $\eta \times 100$ percentage points of χ^2 distribution with mp degrees of freedom, and $\chi_{2s} \equiv \chi^{2s}_{mp}(\eta)/mp(mp+2) \cdots (mp+2s-2)$. Table I and Table II give the values of $T_0^2(\eta)$ calculated by the above formula for $p=3$, 4 and $\eta=0.01$, 0.05.

2.1. *The power of the test based on the statistic* T_0^2. In order to evaluate the power function of the test based on the statistic $\operatorname{tr} \boldsymbol{L}^{-1}\boldsymbol{T}$ or $\operatorname{tr} \boldsymbol{L}^{-1}\boldsymbol{B}$, we shall consider the sampling distribution of the general criterion $T_0^2 = m \operatorname{tr} \boldsymbol{L}^{-1}\boldsymbol{V} = \operatorname{tr} \boldsymbol{L}^{-1}(\sum_{\nu=1}^{m} \boldsymbol{y}_\nu'\boldsymbol{y}_\nu) = \sum_{\nu=1}^{m} \boldsymbol{y}_\nu \boldsymbol{L}^{-1}\boldsymbol{y}_\nu'$ under the hypothesis that $\boldsymbol{y}_\nu$ has the mean vector $\boldsymbol{\eta}_\nu$, all of which are not zero. If we have so large value of n thet $\boldsymbol{L}$ may be replaced by the population covariance matrix Λ, we can consider $\chi_0^2 = \operatorname{tr} \Lambda^{-1}\Big(\sum_{\nu=1}^{m} \boldsymbol{y}_\nu'\boldsymbol{y}_\nu\Big)$ instead of T_0^2. In this case, χ_0^2 is distributed according to the non-central chi-square distribution with mp degrees of freedom and with the parameter δ, where

$$\delta^2 = \sum_{\nu=1}^{m} \boldsymbol{\eta}_\nu \boldsymbol{\Lambda}^{-1}\boldsymbol{\eta}_\nu' , \tag{14}$$

since $\chi_0^2=\sum_{\nu=1}^{m}\chi_{0\nu}^2=\sum_{\nu=1}^{m}\boldsymbol{y}_\nu\Lambda^{-1}\boldsymbol{y}_\nu'$, $\chi_{0\nu}^2$ has the non-central chi-square distribution with p degrees of freedom and with the parameter $\delta_\nu=\{\boldsymbol{\eta}_\nu\Lambda^{-1}\boldsymbol{\eta}_\nu'\}^{\frac{1}{2}}$ and $\chi_{0\nu}^2(\nu=1, 2, \cdots, m)$ are mutually independent.

When n is not so large, the more accurate formula is necessary, which will be considered from now on. From the distribution of χ_0^2, we have

$$P_r\{m \operatorname{tr} \Lambda^{-1}\boldsymbol{V}\leqq 2\xi\}=G_\rho(\xi\,;\,\delta)\,, \tag{15}$$

TABLE Ia $T_0^2(0.05)$ **for** $p=3$

m \ n	10	12	14	16	18	20	22	24	26	28	30
1	15.25	13.35	12.22	11.46	10.93	10.53	10.22	9.98	9.78	9.61	9.47
2	23.32	21.04	19.53	18.47	17.68	17.07	16.59	16.21	15.88	15.61	15.39
3	31.94	28.73	26.61	25.12	24.01	23.16	22.49	21.95	21.50	21.12	20.80
4	40.46	36.28	33.52	31.58	30.14	29.04	28.17	27.47	26.89	26.41	25.99
5	48.44	43.40	40.08	37.74	36.01	34.68	33.63	32.78	32.08	31.50	31.00
6	56.52	50.56	46.64	43.88	41.84	40.27	39.03	38.03	37.21	36.52	35.93
7	64.53	57.66	53.14	49.95	47.60	45.79	44.37	43.22	42.27	41.47	40.79
8	72.50	64.72	59.59	55.98	53.31	51.27	49.65	48.35	47.27	46.37	45.61
9	80.43	71.73	66.00	61.97	58.99	56.70	54.90	53.44	52.24	51.23	50.38
10	88.33	78.72	72.38	67.93	64.63	62.11	60.12	58.51	57.18	55.96	55.12
12	104.0	92.60	85.07	79.77	75.85	72.85	70.47	68.56	66.98	65.65	64.53
14	119.7	106.4	97.69	91.54	87.00	83.51	80.76	78.53	76.70	75.16	73.85
16	135.1	120.1	110.2	103.2	98.08	94.12	90.99	88.46	86.37	84.62	83.13
18	150.5	133.8	122.7	114.9	109.1	104.7	101.2	98.34	95.99	94.03	91.36
20	165.8	147.3	135.1	126.5	120.1	115.2	111.3	108.2	105.6	103.4	101.6

m \ n	35	40	45	50	60	80	100
1	9.18	8.99	8.85	8.74	8.57	8.37	8.26
2	14.94	14.61	14.37	14.17	13.89	13.55	13.35
3	20.18	19.73	19.48	19.11	18.72	18.24	17.97
4	25.19	24.61	24.17	23.82	23.32	22.71	22.36
5	30.03	29.33	28.80	28.38	27.77	27.03	26.61
6	34.79	33.96	33.33	32.85	32.13	31.27	30.76
7	39.48	38.53	37.81	37.25	36.42	35.43	34.85
8	44.12	43.04	42.23	41.59	40.66	39.53	38.88
9	48.72	47.52	46.60	45.89	44.85	43.59	42.86
10	53.28	51.96	50.95	50.16	49.01	47.62	46.81
12	62.34	60.76	59.56	58.62	57.25	55.59	54.63
14	71.31	69.47	68.08	66.99	65.39	63.46	62.34
16	80.23	78.14	76.54	75.30	73.48	71.27	69.99
18	89.11	86.75	84.97	83.57	81.52	79.04	77.59
20	97.95	95.34	93.35	91.80	89.52	86.76	85.15

where $\rho \equiv mp/2$ and

$$G_\rho(\xi\,;\,\delta)=e^{-\frac{\delta^2}{2}}\sum_{j=0}^{\infty}\frac{(\delta^2/2)^j}{j!\Gamma(\rho+j)}\int_0^\xi t^{\rho+j-1}e^{-t}\,dt\,. \tag{16}$$

According to the Welch-James' method [6, 9] which was used to obtain the formula (13), we try to evaluate

$$P_r\{m\ \mathrm{tr}\ \boldsymbol{L}^{-1}\boldsymbol{V}\leqq 2\xi\}=\Big[1+\frac{1}{n}\sum_{rstu}\lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu}$$

TABLE Ib $\quad T_0^2(0.01)$ **for** $p=3$

m \ n	10	12	14	16	18	20	22	24	26	28	30
1	28.47	23.59	20.83	19.08	17.86	16.97	16.30	15.76	15.33	14.98	14.68
2	36.80	32.40	29.52	27.50	26.01	24.88	23.99	23.27	22.68	22.18	21.76
3	48.24	42.35	38.49	35.80	33.82	32.32	31.13	30.17	29.39	28.73	28.18
4	59.19	51.85	47.05	43.70	41.24	39.37	37.90	36.72	36.24	35.43	34.74
5	69.88	61.10	55.37	51.37	48.44	46.21	44.46	43.05	41.89	40.92	40.11
6	80.41	70.20	63.55	58.90	55.50	52.91	50.87	49.24	47.90	46.78	45.83
7	90.83	79.20	71.62	66.33	62.46	59.51	57.19	55.33	53.81	52.53	51.46
8	101.2	88.11	79.61	73.69	69.34	66.04	63.44	61.35	59.64	58.22	57.01
9	111.5	96.98	87.56	80.99	76.17	72.51	69.63	67.32	65.42	63.84	62.50
10	121.7	105.8	95.45	88.24	82.96	78.94	75.78	73.24	71.16	69.42	67,95
12	142.1	123.4	111.2	102.7	96.49	91.74	88.02	85.03	82.57	80.52	78.79
14	162.2	140.8	126.8	117.0	109.8	104.4	100.1	96.65	93.82	91.47	89.47
16	182.2	158.1	142.2	131.1	123.1	116.9	112.1	108.2	105.0	102.3	100.1
18	202.0	175.1	157.6	145.3	136.3	129.4	124.0	119.7	116.1	113.1	110.6
20	221.5	192.0	172.7	159.2	149.3	141.8	135.8	131.1	127.1	123.8	121.1

m \ n	35	40	45	50	60	80	100
1	14.05	13.67	13.39	13.16	12.83	12.44	12.21
2	20.95	20.37	19.93	19.58	18.84	18.24	17.89
3	27.11	26.33	25.75	25.29	24.63	23.83	73.37
4	32.92	31.96	31.24	30.68	29.86	28.88	28.31
5	38.53	37.39	36.53	35.87	34.90	33.73	33.06
6	44.00	42.68	41.69	40.92	39.79	38.45	37.67
7	49.37	47.88	46.75	45.87	44.60	43.07	42.19
8	54.68	53.00	51.74	50.75	49.32	47.61	46.63
9	59.92	58.06	56.66	55.57	53.99	52.09	51.00
10	65.12	63.08	61.54	60.35	58.61	56.53	55.33
12	75.44	73.03	71.22	69.81	67.75	65.30	63.88
14	85.61	82.84	80.75	79.12	76.75	73.91	72.28
16	95.71	92.56	90.19	88.35	85.66	82.44	80.58
18	105.7	102.2	99.57	97.51	94.50	90.89	88.81
20	115.7	111.8	108.9	106.6	103.3	99.26	96.95

$$(17)\qquad +\frac{1}{n^2}\left\{\frac{4}{3}\sum_{rstuvw}\lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}+\frac{1}{2}\sum_{rstuvwxy}\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\right\}$$
$$+0(n^{-3})\Big]P_r\{m\ \mathrm{tr}\ \Lambda^{-1}V\leqq 2\xi\}\ ,$$

where $\partial_{rs}=\frac{1}{2}(1+\partial_{rs})\frac{\partial}{\partial\lambda_{rs}}$, ($\delta_{rs}$ is Kronecker's δ), and summations are taken over $r,\ s,\ \cdots=1,\ 2,\ \cdots p$.

2.2. *The evaluation of the derivatives of* $P_r\{m\ \mathrm{tr}\Lambda^{-1}V\leqq 2\xi\}$. We

TABLE IIa $T_0^2(0.05)$ **for** $p=4$

m \ n	10	12	14	16	18	20	22	24	26	28	30
1	23.54	19.38	17.09	15.65	14.67	13.95	13.41	12.98	12.64	12.36	12.12
2	32.85	29.04	26.55	24.80	23.51	22.53	21.75	21.13	20.62	20.19	19.82
3	45.11	39.80	36.32	33.89	32.10	30.73	29.65	28.79	28.08	27.48	26.98
4	56.96	50.17	45.74	42.63	40.35	39.61	37.24	36.14	35.24	34.48	33.84
5	68.59	60.34	54.95	51.19	48.42	46.31	44.65	43.32	42.22	41.30	40.52
6	80.09	70.39	64.05	59.62	56.37	53.89	51.94	50.37	49.08	48.01	47.10
7	91.51	80.35	73.07	67.98	64.24	61.39	59.15	57.35	55.87	54.63	53.59
8	102.94	90.32	82.07	76.32	72.09	68.87	66.34	64.30	62.63	61.23	60.05
9	114.4	100.3	91.08	84.64	79.92	76.33	73.50	71.22	69.36	67.80	66.48
10	125.3	109.8	99.72	92.67	87.49	83.54	80.43	77.94	75.89	74.17	72.72
12	147.9	129.5	117.5	109.1	102.9	98.20	94.51	91.53	89.09	87.06	85.33
14	170.3	148.9	135.0	125.2	118.2	112.7	108.5	105.0	102.2	99.84	97.85
16	192.5	168.3	152.5	141.5	133.4	127.2	122.4	118.5	115.2	113.6	110.3
18	214.6	187.6	170.0	157.6	148.6	141.6	136.2	131.8	128.2	125.2	122.7
20	236.6	206.8	187.3	173.7	163.7	156.0	150.0	145.2	141.2	137.9	135.0

m \ n	35	40	45	50	60	80	100
1	11.62	11.32	11.09	10.92	10.66	10.35	10.16
2	19.12	18.61	18.22	17.92	17.48	16.95	16.65
3	26.00	25.29	24.76	24.35	23.74	23.01	22.59
4	32.60	31.70	31.03	30.50	29.73	28.81	28.28
5	39.02	37.94	37.12	36.49	35.56	34.44	33.80
6	45.33	44.06	43.11	42.36	41.27	39.97	39.21
7	51.56	50.11	49.01	48.15	46.90	45.41	44.54
8	57.76	56.11	54.87	53.90	52.50	50.81	49.83
9	63.92	62.08	60.70	59.62	58.05	56.16	55.07
10	69.92	67.90	66.38	65.19	63.47	61.39	60.20
12	82.00	79.60	77.79	76.38	74.32	71.86	70.44
14	93.98	91.20	89.10	87.47	85.09	82.23	80.58
16	105.89	102.73	100.34	98.48	95.77	92.52	90.64
18	117.8	114.2	111.5	109.4	106.4	102.8	100.6
20	129.6	125.6	122.7	120.4	117.0	112.9	110.6

consider

$$J = P_r\{m \operatorname{tr}(\Lambda+\varepsilon)^{-1}V \leqq 2\xi\} , \tag{18}$$

where ε is a symmetric matrix composing of the small increments ε_{ij} to λ_{ij}. Then, by Taylor's theorem.

$$J = \left[1 + \sum_{rs} \varepsilon_{rs}\partial_{rs} + \frac{1}{2}\sum_{rstu} \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} + \cdots\right] P_r\{m \operatorname{tr}\Lambda^{-1}V \leqq 2\xi\} . \tag{19}$$

On the other hand we can also express J in the form

TABLE IIb $T_0^2(0.01)$ **for** $p=4$

m \ n	10	12	14	16	18	20	22	24	26	28	30
1	44.84	34.25	28.86	25.63	23.49	21.97	20.84	19.97	19.28	18.71	18.25
2	50.26	43.42	48.98	35.89	33.63	31.92	30.57	29.49	28.61	27.87	27.25
3	66.01	56.94	51.06	46.98	43.99	41.73	39.96	38.54	37.37	36.40	35.59
4	81.09	69.86	62.59	57.55	53.87	51.07	48.89	47.13	45.70	44.51	43.50
5	95.81	82.47	73.83	67.84	63.46	60.15	57.55	55.47	53.77	52.36	51.17
6	110.33	94.88	84.88	77.94	72.88	69.05	66.05	63.65	61.68	60.05	58.67
7	124.70	107.15	95.80	87.92	82.18	77.83	74.43	72.70	69.47	67.62	65.06
8	139.17	119.49	106.76	97.93	91.50	86.62	82.81	79.75	77.26	75.18	73.43
9	153.44	131.65	117.56	107.79	100.67	65.28	91.06	87.68	84.91	82.62	80.68
10	167.31	143.50	128.09	117.41	109.63	103.73	99.11	95.42	92.40	89.88	87.76
12	195.55	167.56	149.45	136.89	127.74	120.80	115.37	111.03	107.38	104.52	102.03
14	223.51	191.39	170.60	156.19	145.67	137.70	147.47	126.47	122.39	118.99	116.13
16	251.29	215.08	191.63	175.37	163.50	154.50	147.46	141.82	137.21	133.37	130.13
18	278.89	238.63	212.55	194.45	181.24	171.22	163.38	157.09	151.95	147.67	144.06
20	306.18	261.96	233.29	213.39	198.85	187.82	179.18	172.26	166.59	161.87	157.89

m \ n	35	40	45	50	60	80	100
1	17.22	16.65	16.23	15.92	15.44	14.85	14.51
2	26.05	25.19	24.54	24.03	23.31	22.44	21.93
3	34.01	32.88	32.03	31.37	30.41	29.27	28.62
4	41.56	40.17	39.13	38.32	37.14	35.74	34.94
5	48.87	47.22	45.99	45.02	43.64	41.98	41.04
6	56.01	54.11	52.69	51.58	49.98	48.07	46.98
7	63.04	60.89	59.27	58.02	56.21	54.04	52.80
8	70.05	67.64	65.53	64.13	62.09	59.67	58.29
9	76.95	74.28	72.28	70.72	68.47	65.80	64.26
10	83.68	80.76	78.58	76.88	74.42	71.49	69.81
12	97.23	93.79	91.22	89.22	86.33	82.88	80.91
14	110.6	106.7	103.7	101.4	98.07	94.12	91.83
16	123.9	119.4	116.1	113.5	109.7	105.2	102.6
18	137.1	132.1	128.4	125.5	121.3	116.8	113.4
20	150.2	144.7	140.6	137.4	133.7	127.2	124.0

$$(20)\qquad J=(2\pi)^{-p}|\Lambda|^{-\frac{m}{2}}\int_R \exp\left\{-\frac{1}{2}\sum_{\alpha=1}^{m}(\boldsymbol{y}_\alpha-\boldsymbol{\eta}_\alpha)\Lambda^{-1}(\boldsymbol{y}_\alpha-\boldsymbol{\eta}_\alpha)'\right\}\prod_\alpha d\boldsymbol{y}_\alpha\ ,$$

where R is the domain defined by

$$(21)\qquad R:\sum_{\alpha=1}^{m}\boldsymbol{y}_\alpha(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{y}_\alpha'\leqq 2\xi\ .$$

Now we make the non-singular linear transformations

$$\boldsymbol{y}_\alpha=\boldsymbol{Z}_\alpha\boldsymbol{C}\ ,\qquad \boldsymbol{\eta}_\alpha=\boldsymbol{\zeta}_\alpha\boldsymbol{C}\ ,\qquad \alpha=1,2,\cdots,m$$

such that

$$\frac{1}{2}\boldsymbol{C}(\Lambda+\varepsilon)^{-1}\boldsymbol{C}'=\boldsymbol{I}\ ,\qquad \frac{1}{2}\boldsymbol{C}\Lambda^{-1}\boldsymbol{C}'=\boldsymbol{I}-\gamma\ ,$$

where γ is a diagonal matrix, diag $\{\gamma_1,\gamma_2,\cdots,\gamma_p\}$. Under these transformations J becomes

$$(22)\qquad J=\pi^{-p}|\boldsymbol{I}-\gamma|^{-\frac{m}{2}}\int_{R'}\exp\left\{-\sum_{\alpha=1}^{m}(\boldsymbol{Z}_\alpha-\zeta_\alpha)(\boldsymbol{I}-\gamma)(\boldsymbol{Z}_\alpha-\zeta_\alpha)'\right\}\prod_\alpha d\boldsymbol{Z}_\alpha$$

$$R':\sum_{\alpha=1}^{m}\boldsymbol{Z}_\alpha\boldsymbol{Z}_\alpha'\leqq\xi\ .$$

Furthermore this integration can be expressed in the very simple form as below:

$$J=\pi^{-p}|\boldsymbol{I}-\gamma|^{\frac{m}{2}}\int_{R'}\exp\left\{-\sum_{\alpha=1}^{m}\boldsymbol{Z}_\alpha\boldsymbol{Z}_\alpha'+\sum_{\alpha=1}^{m}\boldsymbol{Z}_\alpha\gamma\boldsymbol{Z}_\alpha'\right.$$

$$\left.+2\sum_{\alpha=1}^{m}\boldsymbol{Z}_\alpha(\boldsymbol{I}-\gamma)\zeta_\alpha-\sum_{\alpha=0}^{m}\zeta_\alpha(\boldsymbol{I}-\gamma)\zeta_\alpha\right\}\prod_\alpha d\boldsymbol{y}_\alpha$$

$$=\pi^{-p}|\boldsymbol{I}-\gamma|^{\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{\nu_{11},\cdots,\nu_{mp}=0}^{\infty}\ \sum_{\mu_{11},\cdots,\mu_{mp}=0}^{\infty}\frac{\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\gamma_i^{\nu_{\alpha i}}(1-\gamma_i)^{\mu_{\alpha i}}2^{\mu_{\alpha i}}\xi_{\alpha i}^{\mu_{\alpha i}}}{\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\nu_{\alpha i}!\ \mu_{\alpha i}!}$$

$$\times\int_{R'}\prod_{\alpha=1}^{m}\prod_{i=1}^{p}Z_{\alpha i}^{2\nu_{\alpha i}+\mu_{\alpha i}}e^{-\sum_{\alpha=1}^{m}\sum_{i=1}^{p}Z_{i\alpha}^2}\prod_{\alpha=1}^{m}\prod_{i=1}^{p}dZ_{\alpha i}$$

$$=\pi^{-p}|\boldsymbol{I}-\gamma|^{\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{\nu_{11},\cdots,\nu_{mp}=0}^{\infty}\ \sum_{\mu_{11},\cdots,\mu_{mp}=0}^{\infty}\prod_{\alpha=1}^{m}\prod_{i=0}^{p}\frac{\gamma_i^{\nu_{\alpha i}}(1-\gamma_i)^{2\mu_{\alpha i}}2^{2\mu_{\alpha i}}\xi^{2\mu_{\alpha i}}}{\nu_{\alpha i}!\ (2\mu_{\alpha i})!}$$

$$\times\int_{\sum_{\alpha=1}^{m}\sum_{i=1}^{p}t_{\alpha i}\leqq\xi}\prod_{\alpha=1}^{m}\prod_{i=1}^{p}t_{\alpha i}^{\nu_{\alpha i}+\mu_{\alpha i}-\frac{1}{2}}e^{-\sum_{\alpha=1}^{m}\sum_{i=1}^{p}t_{\alpha i}}\prod_{\alpha,i}dt_{\alpha i}$$

$$=|\boldsymbol{I}-\gamma|^{\frac{m}{2}}e^{-\frac{\delta^2}{3}}\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\sum_{\nu_{\alpha i},\mu_{\alpha i}=0}^{\infty}\frac{\gamma_i^{\nu_{\alpha i}}(1-\gamma_i)^{2\mu_{\alpha i}}\xi_{\alpha i}^{2\mu_{\alpha i}}}{\nu_{\alpha i}!\ \mu_{\alpha i}!}\ \frac{\Gamma\left(\nu_{\alpha i}+\mu_{\alpha i}+\frac{1}{2}\right)}{\Gamma\left(\mu_{\alpha i}+\frac{1}{2}\right)}$$

$$\times \frac{1}{\Gamma\left[\sum_{\alpha,i}(\nu_{\alpha i}+\mu_{\alpha i})+\rho\right]}\int_0^\xi t^{\sum_{\alpha,i}(\nu_{\alpha i}+\mu_{\alpha i})+\rho-1}e^{-t}\,dt$$

$$=|\boldsymbol{I}-\gamma|^{\frac{m}{2}}e^{-\frac{\delta^2}{2}}\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\sum_{\mu_{\alpha i}=0}^{\infty}\frac{(1-\gamma_i)^{2\mu_{\alpha i}}\xi^{2\mu_{\alpha i}}}{\mu_{\alpha i}!}$$

$$\times\sum_{\nu_{\alpha i}=0}^{\infty}\frac{\gamma_i^{\nu_{\alpha i}}!}{\nu_{\alpha i}!}\left(\nu_{\alpha i}+\mu_{\alpha i}-\frac{1}{2}\right)\left(\nu_{\alpha i}+\mu_{\alpha i}-\frac{3}{2}\right)\cdots\left(\mu_{\alpha i}+\frac{1}{2}\right)G_{\sum(\nu_{\alpha i}+\mu_{\alpha i})+\rho}(\xi)\,,$$

where $G_a(\xi)=[\Gamma(a)]^{-1}\int_0^\xi t^{a-1}e^{-t}\,dt$. If we define a operator E for fixed ξ and for any positive integer f such that $E^fG_\rho(\xi)=G_{\rho+f}(\xi)$, we can write

$$\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\sum_{\nu_{\alpha i}=0}^{\infty}\frac{\gamma_i^{\nu_{\alpha i}}}{\nu_{\alpha i}!}\left(\nu_{\alpha i}+\mu_{\alpha i}-\frac{1}{2}\right)\left(\nu_{\alpha i}+\mu_{\alpha i}-\frac{3}{2}\right)\cdots\left(\mu_{\alpha i}+\frac{1}{2}\right)G_{\sum(\nu_{\alpha i}+\mu_{\alpha i})+\rho}(\xi)$$

$$=\prod_{\alpha=1}^{m}\prod_{i=1}^{p}(1-\gamma_iE)^{-(\mu_{\alpha i}+\frac{1}{2})}G_{\sum\mu_{\alpha i}+\rho}(\xi)\,.$$

Therefore

$$J=\left\{\frac{|\boldsymbol{I}-\gamma E|}{|\boldsymbol{I}-\gamma|}\right\}^{-\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{\mu_{11},\cdots,\mu_{mp}}^{\infty}\frac{\prod_{\alpha=1}^{m}\prod_{i=1}^{p}(1-\gamma_i)^{2\mu_{\alpha i}}\zeta_{\alpha i}^{2\mu_{\alpha i}}(1-\gamma_iE)^{-\mu_{\alpha i}}}{\prod_{\alpha=1}^{m}\prod_{i=1}^{p}\mu_{\alpha i}!}G_{\sum\mu_{\alpha i}+\rho}(\xi)$$

$$(27)\quad =\left\{\frac{|\boldsymbol{I}-\gamma E|}{|\boldsymbol{I}-\gamma|}\right\}^{-\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{j=0}^{\infty}\frac{1}{j!}\left[\sum_{\alpha=1}^{m}\sum_{i=1}^{p}(1-\gamma_i)^2\zeta_{\alpha i}^2(1-\gamma_iE)^{-1}\right]^jG_{\rho+j}(\xi)$$

$$=\left\{\frac{|\boldsymbol{I}-\gamma E|}{|\boldsymbol{I}-\gamma|}\right\}^{-\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{j=0}^{\infty}\frac{1}{j!}\left[\sum_{\alpha=1}^{m}\zeta_\alpha(\boldsymbol{I}-\gamma)(\boldsymbol{I}-\gamma E)^{-1}(\boldsymbol{I}-\gamma)\zeta_\alpha'\right]^jG_{\rho+j}(\xi)\,.$$

Since

$$(24)\quad \frac{|\boldsymbol{I}-\gamma E|}{|\boldsymbol{I}-\gamma|}=\frac{|\Lambda^{-1}-[(\Lambda+\varepsilon)^{-1}-\Lambda^{-1}]\Delta|}{|\Lambda^{-1}|}=|\boldsymbol{I}-[(\Lambda+\boldsymbol{\varepsilon})^{-1}\Lambda-\boldsymbol{I}]\Delta|$$

and

$$(25)\quad \zeta_\alpha(\boldsymbol{I}-\gamma)(\boldsymbol{I}-\gamma E)^{-1}(\boldsymbol{I}-\gamma)\zeta_\alpha'=\frac{1}{2}\eta_\alpha[\boldsymbol{I}-\{(\Lambda+\varepsilon)^{-1}\Lambda-\boldsymbol{I}\}\Delta]^{-1}\Lambda^{-1}\eta_\alpha'\,,$$

we have the expression

$$(26)\quad J=\{|\boldsymbol{I}-\boldsymbol{X}\Delta|\}^{-\frac{m}{2}}e^{-\frac{\delta^2}{2}}\sum_{j=0}^{\infty}\frac{1}{j!}\left[\frac{1}{2}\sum_{\alpha=1}^{m}\eta_\alpha(\boldsymbol{I}-\boldsymbol{X}\Delta)^{-1}\Lambda^{-1}\eta_\alpha'\right]^jG_{\rho+j}(\xi)\,,$$

where $\Delta\equiv E-1$ and $\boldsymbol{X}=(\Lambda+\boldsymbol{\varepsilon})^{-1}\Lambda-\boldsymbol{I}$.

In order to obtain the derivatives of $P_r\{m\,\mathrm{tr}\,\Lambda^{-1}\boldsymbol{V}\leqq 2\xi\}$, we must carry out the expansion of the above J in powers of ε_{rs} and compare the result of the expansion with (19). Since

$$\{|\boldsymbol{I}-\boldsymbol{X}\Delta|\}^{-\frac{m}{2}}=1+\frac{m}{2}\operatorname{tr}\boldsymbol{X}\Delta+\frac{m}{4}\Big[\operatorname{tr}(\boldsymbol{X}\Delta)^2+\frac{m}{2}(\operatorname{tr}\boldsymbol{X}\Delta)^2\Big]+\cdots,$$

and $\quad(\boldsymbol{I}-\boldsymbol{X}\Delta)^{-1}=\boldsymbol{I}+\boldsymbol{X}\Delta+(\boldsymbol{X}\Delta)^2+\cdots,$

we have, noting $\delta^2=\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\Lambda^{-1}\boldsymbol{\eta}'_\alpha$

$$\begin{aligned}J=\Big\{&1+\frac{m}{2}\operatorname{tr}\boldsymbol{X}\Delta+\frac{m}{4}\Big[\operatorname{tr}(\boldsymbol{X}\Delta)^2+\frac{m}{2}(\operatorname{tr}\boldsymbol{X}\Delta)^2\Big]+\frac{m}{4}(\operatorname{tr}\boldsymbol{X}\Delta)\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\boldsymbol{X}\Lambda^{-1}\boldsymbol{\eta}'_\alpha(\Delta^2+\Delta)\\ &+\frac{1}{2}\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\boldsymbol{X}\Lambda^{-1}\boldsymbol{\eta}'_\alpha(\Delta^2+\Delta)+\frac{1}{2}\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\boldsymbol{X}^2\Lambda^{-1}\boldsymbol{\eta}'_\alpha(\Delta^3+\Delta^2)\\ &+\frac{1}{8}\Big(\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\boldsymbol{X}\Lambda^{-1}\boldsymbol{\eta}'_\alpha\Big)^2(\Delta^4+2\Delta^3+\Delta^2)\\ &+(\text{terms of higher powers than }\boldsymbol{X}^2)\Big\}G_\rho(\xi\,;\,\delta)\,.\end{aligned}\tag{27}$$

Now we interpret E as the operator such that $E^f G_p(\xi\,;\,\delta)=G_{\rho+f}(\xi\,;\,\delta)$. Let us use the abbreviated notations

$$\Lambda_{rs}=\partial_{rs}\Lambda=\frac{1}{2}(1+\delta_{rs})\frac{\partial}{\partial\lambda_{rs}}\Lambda,\quad [rs]=\operatorname{tr}\Lambda^{-1}\Lambda_{rs},\quad [rs|tu]=\operatorname{tr}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu},\cdots$$

$$\delta^2_{(rs)}=\partial_{rs}\delta^2=\partial_{rs}\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\Lambda^{-1}\boldsymbol{\eta}'_\alpha=\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\boldsymbol{\eta}'_\alpha,$$

$$\delta^2_{(rs,tu)}=\sum_{\alpha=1}^{m}\boldsymbol{\eta}_\alpha\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\boldsymbol{\eta}'_\alpha,\ \cdots.$$

Since $\boldsymbol{X}$ has the expansion

$$\boldsymbol{X}=-\sum_{rs}\varepsilon_{rs}\Lambda^{-1}\Lambda_{rs}+\sum_{rstu}\varepsilon_{rs}\varepsilon_{tu}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}-\sum_{rstuvw}\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\Lambda_{vw}+\cdots$$

J can be written

$$J=\{1-\sum\varepsilon_{rs}(F_1)+\sum\varepsilon_{rs}\varepsilon_{tu}(F_2)-\cdots\}G_p(\xi\,;\,\delta)\,,\tag{28}$$

where

$$(F_1)=\frac{m}{2}[rs]\Delta+\frac{1}{2}\delta^2_{(rs)}(\Delta^2+\Delta)\,,$$

$$\begin{aligned}(F_2)=&\frac{m}{2}[rs|tu]\Big(\Delta+\frac{1}{2}\Delta^2\Big)+\frac{m^2}{8}[rs][tu]\Delta^2+\frac{m}{4}[rs]\delta^2_{(tu)}(\Delta^3+\Delta^2)\\ &+\frac{1}{2}\delta^2_{(rs.tu)}(\Delta^3+2\Delta^2+\Delta)+\frac{1}{8}\delta^2_{(rs)}\delta^2_{(tu)}(\Delta^4+2\Delta^3+\Delta^2)\end{aligned}$$

and so on. Thus we have

$$\partial_{rs}P_r\{m \operatorname{tr} \Lambda^{-1}V \leqq 2\xi\} = -\left\{\frac{m}{2}[rs]\Delta + \frac{1}{2}\delta^2_{(rs)}(\Delta^2+\Delta)\right\}G_\rho(\xi\,;\,\delta)$$
(29)
$$= \left\{\frac{m}{2}[rs] + \frac{1}{2}\delta^2_{(rs)}E^2\right\}g_\rho(\xi\,;\,\delta)\,,$$

$$\partial_{rs}\partial_{tu}P_r\{m \operatorname{tr} \Lambda^{-1}V \leqq 2\xi\} = -\left\{\frac{m}{2}[rs|tu](E^2+E) + \frac{m^2}{4}[rs][tu](E^2-E)\right.$$
(30)
$$+\frac{m}{2}[rs]\delta^2_{(tu)}(E^3-E^2) + \delta^2_{(rs.tu)}E^3$$
$$\left. + \frac{1}{4}\delta^2_{(rs)}\delta^2_{(tu)}(E^4-E^3)\right\}g_\rho(\xi\,;\,\delta)\,,$$

and so on, where

$$g_\rho(\xi\,;\,\delta) = D_\rho(\xi\,;\,\delta) = e^{-\frac{\delta^2}{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{\delta^2}{2}\right)^j}{j\,!}\,\frac{1}{\Gamma(\rho+j)}\,\xi^{\rho+j-1}e^{-\xi}$$

$$E^f g_\rho(\xi\,;\,\delta) = g_{\rho+f}(\xi\,;\,\delta)$$

2.3 *The approximate formula of* $P_r\{m \operatorname{tr} L^{-1}V \leqq 2\xi\}$ *up to the order* n^{-1}. Substituting (30) into (17), we obtain

$$P_r\{m \operatorname{tr} L^{-1}V \leqq 2\xi\} = P_r\{m \operatorname{tr} \Lambda^{-1}V \leqq 2\xi\}$$
$$-\frac{1}{n}\sum_{rstu}\lambda_{ur}\lambda_{ts}\left\{\frac{m}{2}[rs|tu](E^2+E) + \frac{m^2}{4}[rs][tu](E^2-E)\right.$$
$$\left.+\frac{m}{2}[rs]\delta^2_{(tu)}(E^3-E^2) + \delta^2_{(rs.tu)}E^3 + \frac{1}{4}\delta^2_{(rs)}\delta^2_{(tu)}(E^4-E^3)\right\}g_\rho(\xi\,;\,\rho)$$
(31)
$$+0(n^{-2})$$
$$= G_\rho(\xi\,;\,\rho) - \frac{1}{n}\left[\frac{m}{4}p(p+1)(E^2+E) + \frac{m^2}{4}p(E^2-E) + \frac{m}{2}\delta^2(E^3-E^2)\right.$$
$$\left.+(p+1)\delta^2E^3 + \sum_{i,j}\sum_{k,l}\lambda^{ki}\lambda^{lj}\left(\sum_{\alpha=1}^{m}\eta_{i\alpha}\eta_{j\alpha}\right)\left(\sum_{\alpha=1}^{m}\eta_{k\alpha}\eta_{l\alpha}\right)(E^4-E^3)\right]g_\rho(\xi\,;\,\delta)\,,$$
$$+0(n^{-2})\,,$$

since individual terms of the right hand side are easily calculated as

$$\sum_{rstu}\lambda_{ur}\lambda_{st}[rs|tu] = \frac{1}{2}\sum_{rstu}\lambda_{ur}\lambda_{st}(\lambda^{ur}\lambda^{st} + \lambda^{us}\lambda^{rt}) = \frac{1}{2}p(p+1)\,,$$
$$\sum_{rstu}\lambda_{ur}\lambda_{st}[rs][tu] = p\,,$$
$$\sum_{rstu}\lambda_{ur}\lambda_{st}[rs]\delta^2_{(tu)} = \sum_{tu}\lambda_{tu}\sum_{\alpha=1}^{m}\eta_\alpha\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\eta'_\alpha = \sum_{\alpha=1}^{m}\eta_\alpha\Lambda^{-1}\left(\sum_{tu}\lambda_{tu}\Lambda_{tu}\right)\Lambda^{-1}\eta'_\alpha = \delta^2\,,$$

$$\sum_{rstu}\lambda_{ur}\lambda_{st}\delta^2_{(rs,tu)}=\sum_{rstu}\lambda_{ur}\lambda_{st}\sum_{\alpha=1}^{m}\eta_\alpha\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\eta'_\alpha=(p+1)\delta^2\ ,$$

$$\sum_{rstu}\lambda_{ur}\lambda_{st}\delta^2_{(rs)}\delta^2_{(tu)}=\sum_{i,j}\sum_{k,l}\lambda^{ki}\lambda^{lj}\Big(\sum_{\alpha=1}^{m}\eta_{i\alpha}\eta_{j\alpha}\Big)\Big(\sum_{\alpha=1}^{m}\eta_{k\alpha}\eta_{l\alpha}\Big)\ .$$

Then we finally obtain, as the approximation up to order n^{-1},

$$\begin{aligned}P_r\{m\ \mathrm{tr}\ \boldsymbol{L}^{-1}\boldsymbol{V}\leqq 2\xi\} &= G_p(\xi\ ;\ \delta)-\frac{1}{n}\Big[\frac{mp}{4}(p-m+1)g_{p+1}(\xi\ ;\ \delta)\\ &+\frac{m}{2}\Big\{\frac{1}{2}p(p+1)+\frac{mp}{2}-\delta^2\Big\}g_{p+2}(\xi\ ;\ \delta)\\ &+\Big\{\Big(\frac{m}{2}+p+1\Big)\delta^2-\sum_{i,j}\sum_{k,l}\lambda^{ki}\lambda^{lj}\Big(\sum_{\alpha=1}^{m}\eta_{i\alpha}\eta_{j\alpha}\Big)\Big(\sum_{\alpha=1}^{m}\eta_{k\alpha}\eta_{l\alpha}\Big)\Big\}g_{p+3}(\xi\ ;\ \delta)\\ &+\sum_{i,j}\sum_{k,l}\lambda^{ki}\lambda^{lj}\Big(\sum_{\alpha=1}^{m}\eta_{i\alpha}\eta_{j\alpha}\Big)\Big(\sum_{\alpha=1}^{m}\eta_{k\alpha}\eta_{l\alpha}\Big)g_{p+4}(\xi\ ;\ \delta)\Big]+0(n^{-2})\ .\end{aligned}\tag{32}$$

The term of order n^{-2} is extremely complicated and omitted here. From (32), we can approximately evaluate the power $P_r\{m\ \mathrm{tr}\ \boldsymbol{L}^{-1}\boldsymbol{V}>T_0^2(\eta)\}$ under the alternative hypothesis.

It is convenient to notice that when $p=2$,

$$\sum_{i,j}\sum_{k,h}\lambda^{ki}\lambda^{hj}\Big(\sum_{\alpha=1}^{m}\eta_{i\alpha}\eta_{j\alpha}\Big)\Big(\sum_{\alpha=1}^{m}\eta_{k\alpha}\eta_{h\alpha}\Big)=\delta^4-2\Big|\sum_{\alpha=1}^{m}\boldsymbol{\eta}'_\alpha\boldsymbol{\eta}_\alpha\Big|/|\boldsymbol{\Lambda}|\tag{33}$$

and when $m=1$

$$\sum_{i,j}\sum_{k,l}\lambda^{ki}\lambda^{lj}\eta_i\eta_j\eta_k\eta_l=\sum_{i,j}\sum_{k,l}(\eta_k\lambda^{ki}\eta_i)(\eta_l\lambda^{lj}\eta_j)=\delta^4\ .\tag{34}$$

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Folks, J. L. and Graybill, F. A., "Heterogeneity of error variances in a randomized block design." *Biometrika*, Vol. 44, (1957), pp, 275-277.

[2] Graybill, F. A., "Variance heterogeneity in a randomized block design" *Biometrics*, Vol. 10, (1954), pp. 516-520.

[3] Hotelling, H., "The generalization of Student's ratio" *Ann. Math. Stat.* Vol. 2, (1931), pp. 360-378.

[4] Hotelling, H. "A generalized T test and measure of multivariate dispersion" *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probalility*, edited by J. Neyman, University of California Press, (1951), pp. 23-41.

[5] Ito, K., "Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic" *Ann. Math. Stat.* Vol. 27, (1956), pp. 1091-1105.

[6] James, G. S., "Tests of linear hypotheses in univariate and multivariate analysis when the ratios of the population variances are unknown," *Biometrika*, Vol. 41 (1954), pp. 19-43.

[7] Siotani, M., On the distribution of the Hotelling's T^2-statistics" *The Procceedings of the Institute of Statistical Mathematics*, (*in Japanese*), Vol. 4, No. 1, (1956), pp. 33-42.

[8] Siotani, M., "On the distributions of the Hotelling's T^2-statistics" *Annals of the Institute of Statistical Mathematics*, Vol. 8, (1956), pp. 1-14.

[9] Welch, B. L., "The generalization of Student's problem when several different population variances are involved," *Biometrika*, Vol. 34, (1947), pp. 28-35.