ON THE TRAFFIC CONTROL AT AN INTERSECTION CONTROLLED BY THE REPEATED FIXED-CYCLE TRAFFIC LIGHTS

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1. Introduction

When traffic control is required at any crossing of two streets, it will be desirable to have some means which enable us to determine the control method of high efficiency for the crossing. The purpose of the present paper is to determine an optimum method, in some sense, of traffic control at a congested crossroads, which has two flows of vehicles, the one in the north and south direction, the other in the west and east direction, control being made by the repeated fixed-cycle traffic lights. By the control of the repeated fixed-cycle we mean the one, in which a green interval of fixed length is followed by a red interval of fixed length, though the length of the red interval may differ from that of the green one. The whole cycle is repeated indefinitely. In practice there are additional warning intervals between the green and red intervals or between the red and the green. But we need not consider such an additional one, for we may appropriately make this belong to the red or the green one.

First we consider the case where two flows have the same degree of arrivals of vehicles, possible passages per unit time of green period being equal. If the control is inequitable and the green period assigned for S–N flow is much longer than that for W–E flow, for example, a long waiting line will rapidly be formed in W–E flow, while that in S–N flow will remain short. Obviously the admission of less passage in S–N flow and more passage in W–E flow is adequate in this case. Even if the difference of green periods for both flows is not so large, the flow to which shorter green period is assigned will have more chances of forming a long waiting line than the one to which longer green period is assigned. From these points of view it will be natural, in this case, to take the method of equitable control for both flows as the optimum one. On the other hand if one flow has more arrivals of vehicles than the other, the control of longer green period for the former will be more efficient. It is necessary to define, in general case, the concept of efficiency of traffic control more clearly, which was considered rather vaguely in the special cases mentioned above.

In any intersection we may consider that the very long waiting line of a flow causes uncontrollable confusion in the traffic, for the congestion of vehicles over the road capacity will, in the long run, cause the impossible communication of the flow. So, it will be reasonable to set up some levels for the length of waiting line of a flow, which warns the risk of uncontrollable confusion in the traffic of the flow. Now that the aim of traffic control is to prevent the confusion in traffic, it must be carried out so as to attain this level as little as possible. We shall call it the *level of confusion* of the flow. It depends on the capacity of the road on which the flow runs. If the level of confusion in S–N flow under a control by some rule is attained rapidly, while that in W–E flow is attained slowly, then it will be more efficient to allow less passage in S–N flow, more passage in W–E flow being allowed.

Considering the efficiency of traffic control in such a sense, we may conclude that the optimum method of control is the one which balances the times, in which confusion levels of both flows are attained. By taking the meaning of optimum method of control as the above, we treated here the following problem. When some random arrivals, possible passages per unit time of green period, the levels of confusion, for both flows, and the length of one cycle of control are given, how should we determine the green periods T and T^* of both flows, in order to make the control optimum?

2. Formulation of the problem

Let T and T^* be the green periods for S–N and W–E flow, respectively, in any cycle of control. Of course these are non-negative numbers.

Define the variables X_n, Y_n as follows:

$X_n \equiv$ the length of waiting line in S–N flow at the end of the time T^* of n-th cycle

$Y_n \equiv$ the length of waiting line in W–E flow at the end of the time T of $(n+1)$th cycle, $n=1, 2, \cdots$

These are random variables under the assumption on arrivals of vehicles in both flows, which will be described later.

In addition, by X_0 we will denote the length of the waiting line in S–N flow at the start of control, and by Y_0 that in W–E flow at the end of the time T in the 1st cycle. As an initial condition we assume that $X_0=0$ and $Y_0=\lambda(T)$, where $\lambda(T)$ denotes the nearest integer to the mean of the arrival number of vehicles in W–E flow during the time T in the 1st cycle.

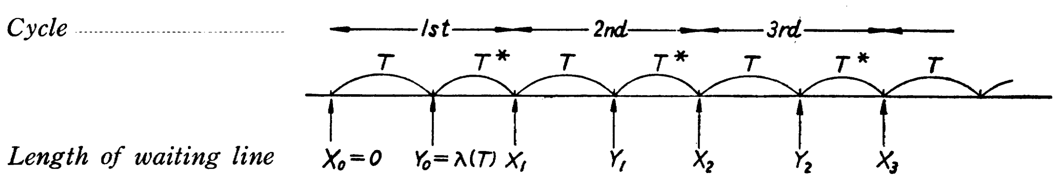

Fig. 1.

Now define the constants

$a \equiv$ the possible passage per unit time of green period in S–N flow

$b \equiv$ the possible passage per unit time of green period in W–E flow

$G \equiv$ the level of confusion of S–N flow

$H \equiv$ the level of confusion of W–E flow

$C \equiv$ the period of one cycle of control.

Of course $C=T+T^*$, the possible passage in green period T in S–N flow is given by aT, and that in green period T^* in W–E flow by bT^*. We substitute aT and bT^* respectively by their nearest integers.

As for the arrivals of vehicles in both flows, we assume either of the following two types of models.

Model 1. Arrivals of vehicles of both flows are independent. In each flow, arrivals during the mutually non-overlapping time intervals are mutually independent random variables, and the distribution of arrival during any time interval depends only on the length of this interval and is irrespective of its position.

When we take the time of start of the control as the time origin and denote respectively by ξ_t and η_t $(t \geqq 0)$ the accumulated numbers of arrivals of S–N and W–E flow from time 0 to time t, the assumption of model 1 means that $\{\xi_t\}$ and $\{\eta_t\}$ are mutually independent and each is a process with independent stationary increments.

If in this model we count arrivals only every unit time starting from the time origin, taking no notice of the manner of arrival during any sub-interval of these unit time, we are led to the following model.

Model 2. Arrivals of vehicles of both flows are independent. When we divide the time axis every unit time starting from the time origin, the arrivals during every unit time are mutually independent random variables having the same distribution.

When model 2 is assumed, it is natural to restrict T and T^* to the multiples of the unit time, and then they may be assumed to be non-negative integers.

Now given T and T^*, divide the time axis alternately at every time T and T^*. Then we represent the distribution of the arrival during each T or T^*-interval as follows:

for S–N flow,

$P_r(k \text{ arrivals})=p_k$ in each T-interval

$P_r(k \text{ arrivals})=p_k^*$ in each T^*-interval;

for W–E flows,

$P_r(k \text{ arrivals})=q_k$ in each T-interval

$P_r(k \text{ arrivals})=q_k^*$ in each T^*-interval;

$k=0, 1, 2, \cdots .$

Here p_k and q_k are functions of T, and p_k^* and q_k^* are functions of T^*. Moreover, we put for convenience $p_k=p_k^*=q_k=q_k^*=0$ for $k=-1, -2, \cdots$.

When a, b, G, H, and C are given, and either of the two models are assumed, we define for any T and T^*, satisfying $T+T^*=C$, the random variables $\boldsymbol{m}^T$ and $\boldsymbol{n}^T$ as follows:

$$\boldsymbol{m}^T=\text{the smallest integer } m \text{ such that } X_m \geqq G ,$$

$$\boldsymbol{n}^T=\text{the smallest integer } n \text{ such that } Y_n \geqq H .$$

Now our problem is stated as follows:

Given a, b, G, H, C and assuming either of the two models, determine T and T^* so that the relation $E[\boldsymbol{m}^T]=E[\boldsymbol{n}^T]$ may hold, where $E[-]$ denotes the expectation.

Of course a and b are non-negative numbers and C is positive number, while G and H are positive integers.

Put $f(T)=E[\boldsymbol{m}^T]$ and $g(T)=E[\boldsymbol{n}^T]$. Then it is clear that $f(T)$ is a monotone increasing function of T, and $g(T)$ is a monotone decreasing function of T. It is intuitively clear that $f(0)<g(0)$ and $f(C)>g(C)$ except in the case where the one traffic is exceedingly busier than the other. So that the problem will have a solution in a practical case. When model 2 is assumed, we restrict T and T^* only to the integral values as mentioned above, but this makes no difficulty in practical purpose.

3. A random walk

Before we proceed further, we consider the following random walk:

A particle moves on a half-line OZ in such a way that, every unit time, it either makes one leap in positive or negative direction or does not move, the possible positions being integral points on OZ; at every leap the particle moves from some integral point on OZ to another. When the particle took the position i after some stage, let p_{ij} be the probability that it takes the position j after the next stage ($i=0, 1, 2, \cdots$; $j=0, 1, 2, \cdots$; $\sum_{j=0}^{\infty} p_{ij}=1$).

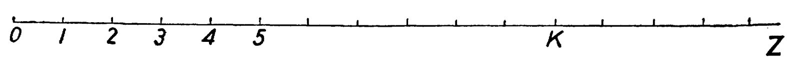

Fig. 2.

Let K be a given positive integer. We denote by $\boldsymbol{n}_i$ the random variable that represents the number of the stage at which the particle, starting from i, gets over the point K for the first time, and put $E[\boldsymbol{n}_i]=m_i$, $i=0, 1, 2, \cdots, K-1$, if these expectations are finite. We shall treat some case, where these are finite, and want to find these quantities in that case.

If we denote by u_{in} the probability that the particle gets over the position K at the nth stage for the first time, starting from i ($n=1, 2, \cdots$; $i=0, 1, 2, \cdots, K-1$), then,

$$(1) \qquad u_{in+1}=\sum_{j=0}^{K-1} u_{jn} p_{ij}, \qquad n=1, 2, \cdots,$$

$$(2) \qquad u_{i1}=1-\sum_{j=0}^{K-1} p_{ij},$$

for $i=0, 1, 2, \cdots, K-1$.

Moreover we denote by $p_{ij}^{(r)}$ the probability that the particle, starting from i, takes the position j after r stages without getting over K all the way, and by $p_i^{(r)}$ the probability that the particle, starting from i, does not get over K during r stages $(i, j=0, 1, 2, \cdots, K-1)$. Clearly $p_i^{(r)}=\sum_{j=0}^{K-1} p_{ij}^{(r)}$.

Then the following theorem holds

THEOREM 1. *If there exist some positive integer r such that*

$$p_i^{(r)}<1, \qquad i=0, 1, 2, \cdots, K-1 \tag{3}$$

then,

$$\sum_{n=1}^{\infty} u_{in}=1, \qquad i=0, 1, 2, \cdots, K-1, \tag{4}$$

that is, the probability is zero that the particle never gets over K starting from i.

Conversely if condition (3) *does not hold, then* (4) *does not hold, and furthermore, for at least one i, $\sum_{n=1}^{\infty} u_{in}$ is equal to zero, that is, the probability is one that the particle never gets over K starting from i.*

PROOF. By $E_i^{(k)}$ we denote the event that the particle, starting from i, does not get over K during k stages, $i=0, 1, 2, \cdots, K-1$; $k=1, 2, \cdots$, and by E_i the one that the particle, starting from i, never gets over K.

Now assume condition (3) for some integer r. Then for an arbitrary positive integer n the occurrence of $E_i^{((n+1)r)}$ implies that of $E_i^{(nr)}$, and the occurrence of the event E_i means the simultaneous occurrence of all $E_i^{(nr)}$. Consequently

$$p_r(E_i)=\lim_{n\to\infty} p_r(E_i^{(nr)})=\sum_{j=0}^{K-1} (\lim_{n\to\infty} p_{ij}^{(nr)}), \; i=0, 1, 2, \cdots, K-1.$$

From the assumption there exists some $0<\varepsilon<1$ such that

$$\sum_{j=0}^{K-1} p_{ij}^{(r)} \leqq 1-\varepsilon, \qquad i=0, 1, 2, \cdots, K-1, \tag{5}$$

and hence

$$p_{ij}^{(2r)}=\sum_{k=0}^{K-1} p_{ik}^{(r)} p_{kj}^{(r)} \leqq \sum_{k=0}^{K-1} p_{ik}^{(r)} \leqq 1-\varepsilon, \qquad i, j=0, 1, 2, \cdots, K-1.$$

Now assume that for n ($\geqq 2$)

$$(6) \qquad p_{ij}^{(nr)} \leqq (1-\varepsilon)^{n-1}, \qquad i, j=0, 1, 2, \cdots, K-1 .$$

Then for $i, j=0, 1, 2, \cdots, K-1$, we have

$$p_{ij}^{((n+1)r)} = \sum_{k=0}^{K-1} p_{ik}^{(r)} p_{kj}^{(nr)} \leqq (1-\varepsilon)^{n-1} \sum_{k=0}^{K-1} p_{ik}^{(r)} \leqq (1-\varepsilon)^n ,$$

therefore by induction (6) holds for every $n \geqq 2$. It follows that $\lim_{n\to\infty} p_{ij}^{(nr)} =0$, and consequently $P_r(E_i)=0$, $i=0, 1, 2, \cdots, K-1$.

Conversely assume that (3) does not hold. Then for any positive integer r there exists some i ($0 \leqq i \leqq K-1$) such that $p_i^{(r)}=1$. It follows that there exist some i ($0 \leqq i \leqq K-1$) and some increasing sequence $\{r_k\}_{k=1,2,\cdots}$ of positive integers such that $p_i^{(r_k)}=1$ for $k=1, 2, \cdots$. Now $P_r(E_i^{(r_k)})=p_i^{(r_k)}$ and in the same way as mentioned above we know that $P_r(E_i)=\lim_{k\to\infty} P_r(E_i^{(r_k)})$. Hence, for some i ($0 \leqq i \leqq K-1$), $P_r(E_i)=1$, that is, the probability is equal to one that the particle never gets over K starting from i.

By this theorem $m_i = \sum_{n=1}^{\infty} n u_{in}$ and more generally $E[\boldsymbol{n}_i^{\mu}] = \sum_{n=1}^{\infty} n^{\mu} u_{in}$ (where μ is a positive integer), $i=0, 2, \cdots, K-1$, under the condition (3).

LEMMA. *Under the condition* (3), *for an arbitrary positive integer* μ, *we have*

$$(7) \qquad \lim_{N\to\infty} N^{\mu} u_{iN} = 0 , \qquad i=0, 1, 2, \cdots, K-1 ,$$

PROOF. From the assumption there exists some positive integer r for which (5) holds. Now take a fixed arbitrary integer ν ($0 \leqq \nu \leqq r-1$). Then

$$u_{i,r+\nu} = \sum_{j=0}^{K-1} p_{ij}^{(r)} u_{j\nu} \leqq \sum_{j=0}^{K-1} p_{ij}^{(r)} \leqq 1-\varepsilon , \qquad i=0, 1, 2, \cdots, K-1 .$$

Assume that for integer k ($k \geqq 1$)

$$(8) \qquad u_{i,kr+\nu} \leqq (1-\varepsilon)^k , \qquad i=0, 1, 2, \cdots , K-1 ,$$

then for $i=0, 1, 2, \cdots, K-1$,

$$u_{i,(k+1)r+\nu} = \sum_{j=0}^{K-1} p_{ij}^{(r)} u_{j,kr+\nu} \leqq (1-\varepsilon)^k \sum_{j=0}^{K-1} p_{ij}^{(r)} \leqq (1-\varepsilon)^{k+1} ,$$

and so by induction (8) holds for every positive integer k. It follows that

$$\lim_{k\to\infty}(kr+\nu)^{\mu}u_{i,kr+\nu}\leqq\lim_{k\to\infty}(kr+\nu)^{\mu}(1-\varepsilon)^{k}=0\ ,$$

$i=0,1,2,\cdots,K-1$, $\nu=0,1,2,\cdots,r-1$, from which we have $\lim_{N\to\infty}N^{\mu}u_{iN}=0$.

From this lemma, the following theorem obviously holds.

Theorem 2. *Under the condition* (3), *the random variable* $\boldsymbol{n}_i$ *has all the moments, that is,* $\sum_{n=1}^{\infty}n^{\mu}u_{in}<\infty$, $\mu=1,2,\cdots$, *for* $i=0,1,2,\cdots,K-1$.

In particular in this case, $m_i=\sum_{n=1}^{\infty}nu_{in}<\infty$, $i=0,1,2,\cdots,K-1$, and we can actually obtain m_i's as follows.

Theorem 3. *If the matrix* I_K-P *is non-singular and condition* (3) *is statisfied, then we have*

$$\begin{pmatrix} m_0 \\ m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_{K-1}\end{pmatrix}=(I_K-P)^{-1}\begin{pmatrix}1\\1\\ \cdot \\ \cdot \\ \cdot \\ 1\end{pmatrix}. \tag{9}$$

where I_K *is a unit matrix of Kth order and*

$$P=\begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0,K-1} \\ p_{10} & p_{11} & \cdots & p_{1,K-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_{K-1,0} & p_{K-1,1} & \cdots & p_{K-1,K-1}\end{pmatrix}$$

Proof. From (1) and (2)

$$\sum_{n=1}^{N}(n+1)u_{i,n+1}+u_{i1}$$
$$=\sum_{j=0}^{K-1}p_{ij}\sum_{n=1}^{N}nu_{jn}+\sum_{j=0}^{K-1}p_{ij}\sum_{n=1}^{N}u_{jn}+\Big(1-\sum_{j=0}^{K-1}p_{ij}\Big)\,,$$
$$i=0,1,2,\cdots,K-1\ .$$

Therefore, $m_{i,N+1}=\sum_{j=0}^{K-1}p_{ij}m_{jN}+c_{iN}$, where $m_{iN}=\sum_{n=1}^{N}nu_{in}$ and $c_{iN}=$

$\sum_{j=0}^{K-1} p_{ij} \sum_{n=1}^{N} u_{jn} + \left(1 - \sum_{j=0}^{K-1} p_{ij}\right)$.

Now let $N \to \infty$, then by theorems 1 and 2 $m_{i,N+1}$, $m_{iN} \to m_i < \infty$ and $c_{iN} \to 1$, and we get $m_i = \sum_{j=0}^{K-1} p_{ij} m_j + 1$, $i = 0, 1, 2, \cdots, K-1$, that is, (9) holds.

We remark that if

$$\text{(10)} \qquad \text{for every } 0 \leqq i \leqq K-1 \text{ there exists } i' > i \text{ such that } p_{ii'} > 0 ,$$

then condition (3) holds.

For we have only to take $r = K$.

Besides as we shall see later (cf. Corollary 3 to Theorem 4) the non-singularity of $I_K - P$ follows from condition (10), so that the assumptions of Theorem 4 are satisfied if condition (10) holds.

For the discrimination of the non-singularity of $I_K - P$ the following criteria will be useful. Let

$$A = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \cdots\cdots \\ \cdots\cdots \\ a_{n1} \cdots a_{nn} \end{pmatrix}$$

be a matrix of nth order with non-negative elements, in which the sum of the elements of every row is not greater than 1. For arbitiary positive integers $i_1, i_2, \cdots, i_k$ $(i_1 < i_2 < \cdots < i_k \leqq n)$ we denote by $A_{i_1 \cdots i_k}$ the submatrix of A obtained by taking its i_1th to i_kth rows and deleting its i_1th to i_kth columns.

We now prove

THEOREM 4. *The following conditions are necessary and sufficient for the matrix $I_n - A$ to be non-singular:*

(1°) *the sum of the elements of at least one of the rows of A is actually less than* 1

(2°) *for every k $(\leqq n-1)$ and for arbitrary positive integers $i_1, i_2, \cdots, i_k$ $(i_1 < i_2 < \cdots < i_k \leqq n)$, $A_{i_1 \cdots i_k}$ cannot be zero-matrix without at least one of the sums of the elements of the i_1th to i_kth rows of it being actually less than* 1.

PROOF. Necessity: We assume that $I_n - A$ is non-singular. Then

(1°) holds obviously. Suppose that (2°) does not hold, and for example, that $A_{1\cdots k}$ be zero matrix for some k ($\leqq n-1$), every sum of the elements of the 1st to kth row of A being equal to 1. Then we have

$$|I_n-A|=\begin{vmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & -a_{k2} & & 1-a_{kk} \end{vmatrix} \begin{vmatrix} 1-a_{k+1,k+2} & -a_{k+1,k+2} & \cdots & -a_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n,k+1} & -a_{n\,k+2} & & 1-a_{n,n} \end{vmatrix} .$$

But it follows from our assumption that the first factor in the right-hand side of the above equality is equal to zero, and so $|I_n-A|=0$ in contradiction to the assumption of non-singularity of I_n-A.

Sufficiency: Suppose that the conditions (1°) and (2°) are satisfied. If I_n-A is singular, there exist n real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, at least one of which is not zero, such that

$$(11)\qquad \begin{cases} \lambda_1(1-a_{11}) & -\lambda_2 a_{12}- & \cdots -\lambda_n a_{1n}=0 \\ -\lambda_1 a_{21} & +\lambda_2(1-a_{21})- & \cdots -\lambda_n a_{2n}=0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\lambda_1 a_{n1} & -\lambda_2 a_{n2}- & \cdots +\lambda_n(1-a_{nn})=0 \end{cases}$$

If necessary by rearranging rows and columns of A we may assume, without loss of generality, that

$$(12)\qquad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$$

In addition we may assume that $\lambda_1>0$, for if $\lambda_1\leqq 0$, we may take $-\lambda_1$, $-\lambda_2, \cdots, -\lambda_n$ instead of $\lambda_1, \lambda_2, \cdots, \lambda_n$. From (11) we get

$$\lambda_1=\lambda_1 a_{11}+\lambda_2 a_{12}+\cdots+\lambda_n a_{1n}$$

Dividing the both sides by λ_1 we have

$$(13)\qquad \begin{aligned} 1&=a_{11}+\frac{\lambda_2}{\lambda_1}a_{12}+\cdots+\frac{\lambda_n}{\lambda_1}a_{1n} \\ &\leqq a_{11}+a_{12}+\cdots+a_{1n}\leqq 1 . \end{aligned}$$

Consequently $a_{11}+\frac{\lambda_2}{\lambda_1}a_{12}+\cdots+\frac{\lambda_n}{\lambda_1}a_{1n}=a_{11}+a_{12}+\cdots a_{1n}$, that is

$$\left(1-\frac{\lambda_2}{\lambda_1}\right)a_{12}+\cdots+\left(1-\frac{\lambda_n}{\lambda_1}\right)a_{1n}=0 .$$

Now in the left-hand side of this expression, each term is non-negative, and so we get

$$(14) \qquad \left(1-\frac{\lambda_2}{\lambda_1}\right)a_{12}=0, \cdots , \qquad \left(1-\frac{\lambda_n}{\lambda_1}\right)a_{1n}=0 .$$

If $a_{12}=a_{13}=\cdots=a_{1n}=0$ we get from assumption (2°) $a_{11}<1$ on the one hand, and get from (13) $a_{11}=1$ on the other hand, which is a contradiction. Therefore, there exists i_1 $(2\leqq i_1\leqq n)$ with $a_{1i_1}\neq 0$. Consequently from (14) $\lambda_1=\lambda_{i_1}$, which means by (12),

$$(15) \qquad \lambda_1=\lambda_2=\cdots=\lambda_{i_1} .$$

If $i_1<n$, we proceed as follows. From (11) we get for every j $(1\leqq j\leqq i_1)$

$$\lambda_j=\lambda_1 a_{j1}+\cdots+\lambda_{i_1}a_{ji_1}+\lambda_{i_1+1}a_{j,i_1+1}+\cdots+\lambda_n a_{jn} .$$

Dividing the both sides of this equation by λ_1 and noticing (15), we get

$$(16) \qquad \begin{aligned} 1&=a_{j1}+\cdots+a_{ji_1}+\frac{\lambda_{i_1+1}}{\lambda_1}a_{j,i_1+1}+\cdots+\frac{\lambda_n}{\lambda_1}a_{jn} \\ &\leqq a_{j1}+\cdots+a_{ji_1}+a_{j,i_1+1}+\cdots+a_{jn}\leqq 1 . \end{aligned}$$

So, in the same way, we have

$$(17) \qquad \left(1-\frac{\lambda_{i_1+1}}{\lambda_1}\right)a_{j,i_1+1}=0, \cdots , \quad \left(1-\frac{\lambda_n}{\lambda_1}\right)a_{jn}=0 .$$

If $a_{j\nu}=0$ for every j and $\nu (1\leqq j\leqq i_1,\ i_1+1\leqq \nu\leqq n)$, we learn from assumption (2°) that at least one of $a_{j1}+\cdots+a_{ji_1}$ $(j=1, \cdots, i_i)$ is less than 1, on the one hand, and from (16) that $a_{j1}+\cdots+a_{ji_1}=1$, $j=1, \cdots, i_1$ on the other hand, which is a contradiction. Consequently there exist some j and i_2 $(1\leqq j\leqq i_i,\ i_i<i_2\leqq n)$, such that $a_{ji_2}\neq 0$. Therefore from (17) we get $\lambda_{i_2}=\lambda_1$, and from (12)

$$(18) \qquad \lambda_1=\cdots=\lambda_{i_2} .$$

In the same way there exists i_3 $(i_2<i_3\leqq n)$ such that $\lambda_1=\cdots=\lambda_{i3}$, and i_4 $(i_3<i_4\leqq n)$ suce that $\lambda_1=\cdots=\lambda_{i_4}$, and so on, as long as $i_2<n$, $i_3<n, \cdots$.

Thus we have finally

$$(19) \qquad \lambda_1=\lambda_2=\cdots=\lambda_n$$

so that from (11)

$$\begin{cases} \lambda_1=\lambda_1(a_{11}+a_{12}+\cdots+a_{1n}) \\ \quad\vdots \qquad\quad \vdots \\ \lambda_1=\lambda_1(a_{n1}+a_{n2}+\cdots+a_{nn}) . \end{cases}$$

But $\lambda_1 \neq 0$, therefore we have

$$\begin{cases} a_{11}+a_{12}+\cdots+a_{1n}=1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}+a_{n2}+\cdots+a_{nn}=1\,. \end{cases}$$

This contradicts assumption (1°).

From this theorem we get at once the following corollaries.

COROLLARY 1. *If every sum of the elements of the row of* A *is less than* 1, *then* $I_n - A$ *is non-singular.*

COROLLARY 2. *If for every* i $(1 \leqq i \leqq n)$ *the following condition is satisfied, then* $I_n - A$ *is non-singular: either there exists* i' $(i < i' \leqq n)$ *with* $a_{ii'} > 0$ *or the sum of the elements of the ith row in* A *is less than* 1.

In particular we have

COROLLARY 3. *If the condition* (10) *of this section* 3 *is satisfied, then* $I_K - P$ *is non-singular.*

By $A\begin{pmatrix} i_1, \cdots, i_r \\ j_1, \cdots, j_s \end{pmatrix}$ we denote the matrix which is made from A by taking its i_1th to i_rth rows and j_1th to j_sth columns. Then

COROLLARY 4. *If the following condition is satisfied, the matrix* $I_n - A$ *is non-singular: there exists some positive integer* m $(<n)$, *such that every sum of the elements of each of the first* m *rows from* A *is equal to* 1, *and every sum of the elements of each among the rest is less than* 1; *and there exist some positive integer* k $(\leqq m)$, $r_1, \cdots, r_{k-1}$ $(r_1 < \cdots < r_{k-1} < r_k = m)$, *such that in each of matrices* $A\begin{pmatrix} 1, & \cdots, r_1 \\ r_1+1, & \cdots, n \end{pmatrix}$, $\cdots$, $A\begin{pmatrix} r_{k-2}+1, \cdots, r_{k-1} \\ r_{k-1}+1, \cdots, n \end{pmatrix}$, $A\begin{pmatrix} r_{k-1}+1, \cdots, r_k \\ r_k+1, \cdots, n \end{pmatrix}$ *there exists at least one non-zero element in every row of it.* (cf. Fig. 3).

Even though A is not of the form stated in Corollary 4, this corollary is applicable if A can be transformed into that form by those permutations, in which the row- and the column-permutations are performed in all the same way. We will see this procedure in some detail.

Let in each of some m $(\leqq n)$ rows of A the sum of its elements be equal to 1 and in each of the rest the sum of its elements be less than 1. First we distinguish two cases. If $m=n$, $I_n - A$ is singular, and if

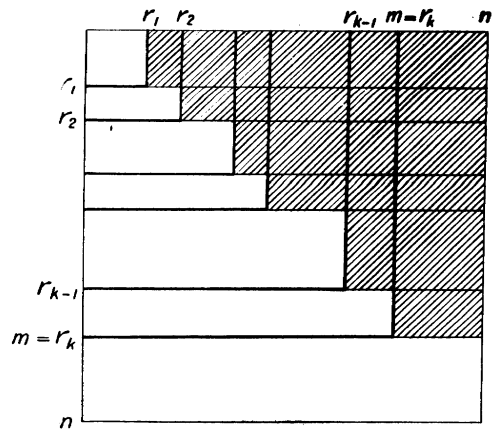

Fig. 3.

$m<n$, we transform A by the above operations into the form, in which the first m rows are such that in each of them sum of its elements is equal to 1. We denote the transformed matrix again by A. Here we distinguish three cases. If $A\begin{pmatrix} 1, \cdots, m \\ m+1, \cdots, n \end{pmatrix}$ is zero-matrix, then it is easily seen that $I_n - A$ is singular. Otherwise we examine the rows of $A\begin{pmatrix} 1, \cdots, m \\ m+1, \cdots, n \end{pmatrix}$. If in each row of this matrix there is at least one non-zero element, then A have the form stated in Corollary 4, and so $I_n - A$ is non-singular. If not we transform A, by carrying out the above operations for 1st to mth rows and columns, into B in such a way that in $B\begin{pmatrix} 1, \cdots, m \\ m+1, \cdots, n \end{pmatrix}$ all the rows, having zero-elements only, are arranged from the top. We denote B newly by A. Let r_{k-1} be the positive integer such that $A\begin{pmatrix} 1, \cdots, r_{k-1} \\ m+1, \cdots, n \end{pmatrix}$ is zero-matrix and in each row of $A\begin{pmatrix} r_{k-1}+1, \cdots, m \\ m+1, \cdots, n \end{pmatrix}$ there is at least one non-zero element. Here we also distinguish three cases according to the form of $A\begin{pmatrix} 1, \cdots, r_{k-1} \\ r_{k-1}+1, \cdots, m \end{pmatrix}$ and proceed as mentioned above, and so on. Thus either we know at some stage of the procedure that $I_n - A$ is singular, or A can be transformed by the above-mentioned operations to the matrix of the form illustrated by Fig. 4 and we know that $I_n - A$ is non-singular, where in Fig. 4 each of the submatrices exhibited by the

shadowed rectangles is the one, in every row of which there exists at least one non-zero element. Thus we have proved the following equivalent to Theorem 4.

COROLLARY 5. *For the non-singularity of the matrix $I_n - A$ it is necessary and sufficient that by the above-mentioned procedure A can be transformed into the form illustrated by Fig. 4, where each of the shadowed rectangles represents such a submatrix that we mentioned above.*

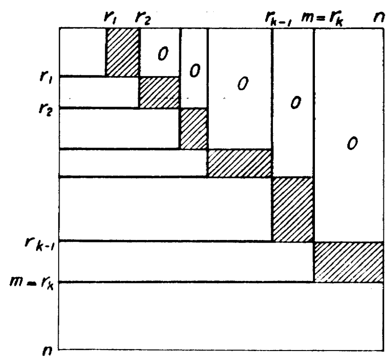

Fig. 4.

4. The waiting line as the random walk mentioned above

The length $\{X_n\}$ or $\{Y_n\}$ of waiting line of S–N or W–E flow can be regarded as the random walk that was mentioned in the preceding section.

For example, take the S–N flow, and fix T and T^*. We may consider that, $X_0=0$ means that the particle lies at 0 at the beginning, and that $X_1, X_2, \cdots$ represent the positions of the particle respectively after the 1st, 2nd, $\cdots$ stage. Let x_n and x_n' be the random variables which denote the arrivals of vehicles in S–N flow respectively during the time T and T^* in the nth cycle. Then by the assumption $x_1, x_2, \cdots, x_1', x_2', \cdots$ are the sequence of mutually independent random variables, and

(20) $$X_n = (X_{n-1} + x_n - aT)^+ + x_n', \qquad n=1, 2, \cdots,$$

where $(X_{n-1} + x_n - aT)^+$ denotes the maximum $[X_{n-1} + x_n - aT, 0]$. From (20) it is obvious that x_n, x_n' are stochastically independent of $X_0, X_1, \cdots, X_{n-1}$. From this fact, together with relation (20), it follows that

$$\begin{aligned} &P_r(X_n = j / X_0 = i_0, \cdots, X_{n-2} = i_{n-2}, X_{n-1} = i) \\ &= P_r(X_n = j / X_{n-1} = i) \\ &= P_r\{(i + x_n - aT)^+ + x_n' = j\} \end{aligned}$$

for arbitrary non-negative integers $i_0, \cdots, i_{n-2}$, i and j. Here it is obvious that $P_r(X_n=j/X_{n-1}=i)$ does not depend on n. These show that $\{X_n\}$ forms the random walk mentioned in Section 3, and that the transition probabilities ${}_{T1}p_{ij}$ are given by

$$(21) \qquad {}_{T1}p_{ij}=P_r\{(i+x_n-aT)^+ + x_n'=j\} \ .$$

In the same way $\{Y_n\}$ forms the random walk mentioned above, and if we denote by y_n and y_n' the arrivals of vehicles in W–E flow respectively during the time T in the nth cycle and the time T in the $(n+1)$th cycle, the transition probabilities ${}_{T2}p_{ij}$ are given by

$$(22) \qquad {}_{T2}p_{ij}=P_r\{(i+y_n-bT^*)^+ + y_n'=j\} \ .$$

Now put ${}_{T1}P=({}_{T1}p_{ij})_{i,j=0,1,\cdots G-1}$, ${}_{T2}P=({}_{T2}p_{ij})_{i,j=0,1,\cdots,H-1}$. If condition (3) holds and if $I_G-{}_{T1}P$ and $I_H-{}_{T2}P$ are non-singular for all T, then by Theorem 4 the functions $f(T)$ and $g(T)$, defined in Section 2, are given as follows:

$f(T)=$the sum of elements of the first row of $(I_G-{}_{T1}P)^{-1}$,

$g(T)=$the sum of elements of the $\lambda(T)$th row of $(I_H-{}_{T2}P)^{-1}$.

As remarked in Section 3, the condition (10) is sufficient for the assumptions of Theorem 4 to hold. Therefore if the condition (10) holds both for ${}_{T1}p_{ij}$ and ${}_{T2}p_{ij}$, for all T, then $f(T)$ and $g(T)$ are given as the above, too.

It is difficult to obtain the explicit forms of $f(T)$ and $g(T)$ theoretically, but in practice the curves of these functions may be obtained by numerical computation, and our problem can be solved. For this purpose it is necessary to count ${}_{T1}P$ and ${}_{T2}P$, which will be done in the following.

Using the notation in Section 2, for fixed T and T^*, we have

$$(23) \qquad P_r(x_n=k)=p_k \ , \qquad P_r(x_n'=k)=p_k^* \ ,$$
$$k=0, \pm 1, \pm 2, \cdots ; \quad n=1, 2, \cdots \ .$$

Now fix i. Then from (21) and stochastic independence of x_n and x_n' we have

$$\begin{aligned} {}_{T1}p_{ij} &= P_r\{(x_n+i-aT)^+ + x_n'=j\} \\ &= P_r(x_n<aT-i)P_r\{(x_n+i-aT)^+ + x_n'=j/x_n<aT-i\} \\ &\quad + P_r(x_n\geqq aT-i)P_r\{(x_n+i-aT)^+ + x_n'=j/x_n\geqq aT-i\} \end{aligned}$$

$$=P_r(x_n<aT-i)\,P_r(x_n'=j/x_n<aT-i)$$
$$+P_r(x_n\geqq aT-i)\,P_r(x_n+i-aT+x_n'=j/x_n\geqq aT-i)$$
$$=P_r(x_n<aT-i)\,P_r(x_n'=j)+P_r(x_n+i-aT+x_n'=j,\ x_n\geqq aT-i)\ .$$

From this we obtain, noticing $p_k^*=0$ for $k<0$,

$$(24)\qquad {}_{T1}p_{ij}=\Big(\sum_{k=0}^{aT-i-1}p_k\Big)p_j^*+\sum_{k=0}^{j}p_{aT-i+k}\,p_{j-k}^*\ .$$

For $i\geqq aT$, noticing $p_k=0$ when $k<0$, (24) becomes

$$ {}_{T1}p_{ij}=\begin{cases}\sum_{k=i-aT}^{j}p_{aT-i+k}\,p_{j-k}^*: & j\geqq i-aT\\ 0 \qquad : & j<i-aT\ .\end{cases}$$

From this it is seen that for $i\geqq aT$, ${}_{T1}p_{ij}$ is completely determined when ${}_{T1}p_{aT,j}=\sum_{k=0}^{j}p_k p_{j-k}^*=P_r(x_1+x_1'=j)$, $j=0,1,2,\cdots$, are known, that is, the ith and the following rows of the matrix of transition probabilities are given as follows:

i \ j	0	1	2	3	4	·	·	·	·	·
aT	${}_{Ta1}p_{T,0}$	${}_{T1}p_{aT,1}$	${}_{T1}p_{aT,2}$	${}_{T1}p_{aT,3}$	${}_{T1}p_{aT,4}$	·	·	·	·	·
$aT+1$	0	${}_{T1}p_{aT,0}$	${}_{T1}p_{aT,1}$	${}_{T1}p_{aT,2}$	${}_{T1}p_{aT,3}$	·	·	·	·	·
$aT+2$	0	0	${}_{T1}p_{aT,0}$	${}_{T1}p_{aT,1}$	${}_{T1}p_{aT,2}$	·	·	·	·	·
$aT+3$	0	0	0	${}_{T1}p_{aT,0}$	${}_{T1}p_{aT,1}$	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·

Next consider the case $i<aT$. From (24) we obtain

$$(25)\qquad {}_{T1}p_{i+1,j+1}={}_{T1}p_{ij}+d_{ij}\ ,$$
$$i=0,1,\cdots,aT-2\ ,\quad j=0,1,2,\cdots\ ,$$

where

$$d_{ij}=\Big(\sum_{k=0}^{aT-i-1}p_k\Big)(p_{j+1}^*-p_j^*)\ .$$

Therefore, in this case, the general ${}_{T1}p_{ij}$ is deduced from ${}_{T1}p_{0j}$ and ${}_{T1}p_{i0}$ adding d_{ij} successively, where ${}_{T1}p_{0j}$ and ${}_{T1}p_{i0}$ are as follows:

$${}_{T1}p_{0j}=\Big(\sum_{k=0}^{aT}p_k\Big)p_j^*+\sum_{k=1}^{j}p_{aT+k}\,p_{j-k}^*\ ,\qquad j=0,1,2,\cdots\ ,$$

$$ {}_{T1}p_{i0}=\left(\sum_{k=0}^{aT-i} p_k\right)p_0^* , \qquad i=0,\,1,\,\cdots,\,aT-1 . $$

Deduction of ${}_{T1}p_{ij}$ for $i<aT$ can be described by the following scheme.

$$
\begin{array}{ccccccccccccc}
{}_{T1}p_{00} & & {}_{T1}p_{01} & & {}_{T1}p_{02} & & {}_{T1}p_{03} & & {}_{T1}p_{04} & \cdot & \cdot & \cdot \\
 & \searrow +d_{00} & & \searrow +d_{01} & & \searrow +d_{02} & & \searrow +d_{03} & & & & \\
{}_{T1}p_{10} & & {}_{T1}p_{11} & & {}_{T1}p_{12} & & {}_{T1}p_{13} & & {}_{T1}p_{14} & \cdot & \cdot & \cdot \\
 & \searrow +d_{10} & & \searrow +d_{11} & & \searrow +d_{12} & & \searrow +d_{13} & & & & \\
{}_{T1}p_{20} & & {}_{T1}p_{21} & & {}_{T1}p_{22} & & {}_{T1}p_{23} & & {}_{T1}p_{24} & \cdot & \cdot & \cdot \\
 & \searrow +d_{20} & & \searrow +d_{21} & & \searrow +d_{22} & & \searrow +d_{23} & & & & \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & \cdot & \cdot & \cdot \\
{}_{T1}p_{aT-2,0} & & {}_{T1}p_{aT-2,1} & & {}_{T1}p_{aT-2,2} & & {}_{T1}p_{aT-2,3} & & {}_{T1}p_{aT-2,4} & \cdot & \cdot & \cdot \\
 & \searrow +d_{aT-2,0} & & \searrow +d_{aT-2,1} & & \searrow +d_{aT-2,2} & & \searrow +{}_{aT-2,3} & & & & \\
{}_{T1}p_{aT-1,0} & & {}_{T1}p_{aT-1,1} & & {}_{T1}p_{aT-1,2} & & {}_{T1}p_{aT-1,3} & & {}_{T1}p_{aT-1,4} & \cdot & \cdot & \cdot
\end{array}
$$

In the same way we can count ${}_{T2}P$. For that we have only to substitute bT^* for aT, H for G, q_j^* for p_j and q_j for p_j^* in the above deduction of ${}_{T1}P$.

5. Special cases

Assuming Poisson processes concerning the arrivals of the vehicles in S–N and W–E flow, we get a case of Model 1 for the arrivals of vehicles. Let α and β respectively be the mean arrivals in S–N and W–E flow per unit time. Fix T and T^*. Then using the notation of Section 2, for $k=0, 1, 2, \cdots$, we have

$$ p_k=\frac{e^{-\alpha T}(\alpha T)^k}{k!} , \qquad p_k^*=\frac{e^{-\alpha T^*}(\alpha T^*)^k}{k!} $$

$$ q_k=\frac{e^{\beta T}(\beta T)^k}{k!} , \qquad q_k^*=\frac{e^{-\beta T^*}(\beta T)^k}{k!} , $$

In this case the condition (10) of Section 3 is clearly satisfied for both

${}_{T1}p_{ij}$ and ${}_{T2}p_{ij}$, for every T. Therefore, by Section 4 we can solve our problem in this case.

Next consider a special case of Model 2 for the arrivals of the vehicles in S–N and W–E flows. It is as follows:

In each flow the arrival of vehicles is given by Model 2 with the additional assumption that at most one car arrives during each unit interval of time, the probabilities of one arrival in each interval for S-N and W–E flows being $p(>0)$ and $q(>0)$ respectively. Fix T and T^*. Then using the notation of Section 2, we have

$$p_k=\binom{T}{k}p^k(1-p)^{T-k}, \qquad k=0, 1, \cdots, T$$

$$p_k^*=\binom{T^*}{k}p^k(1-p)^{T^*-k}, \qquad k=0, 1, \cdots, T^*$$

$$q_k=\binom{T}{k}q^k(1-q)^{T-k}, \qquad k=0, 1, \cdots, T$$

$$q_k^*=\binom{T^*}{k}q^k(1-q)^{T^*-k}, \qquad k=0, 1, \cdots, T^* .$$

In the case of this model we shall consider the case where $C\geqq 2$. Then the following theorem holds.

THEOREM 5. *If $a\leqq 1$ and $b\leqq 1$, then condition* (10) *of Section* 3 *is satisfied for both ${}_{T1}p_{ij}$ and ${}_{T1}p_{ij}$, for every integer $0<T<C$.*

PROOF. We remark that $p_k>0$ for $0\leqq k\leqq T$ and $p_k^*>0$ for $0\leqq k\leqq T^*$. Now, from $a\leqq 1$ we have $(i+T-aT)^+=i+(1-a)T$, so that

$$\begin{aligned} {}_{T1}p_{i,i+(1-a)T+1}&=P_r\{(i+x_n-aT)^+ +x_n'=i+(1-a)T+1\} \\ &\geqq P_r(x_n=T,\ x_n'=1)=p_Tp_1^* . \end{aligned}$$

But $T\geqq 1$, $T^*\geqq 1$, and so $p_T>0$, $p_1^*>0$. Consequently ${}_{T1}p_{i,i+(1-a)T+1}>0$. That is, taking $i'=i+(1-a)T+1$, condition (10) holds for ${}_{T1}p_{ij}$. In the same way we can conclude from $b\leqq 1$ that condition (10) is satisfied for ${}_{T2}p_{ij}$.

From Theorem 5 and Section 4 we can solve our problem in this case provided that $a\leqq 1$ and $b\leqq 1$.

Even though a or b is greater than 1, we can ensure the validity of condition (10) by making restriction on the range of variation of T and T^*. That is

THEOREM 6. *Let λ $(0<\lambda<C)$ and μ $(0<\mu<C)$ be such given integers that $\lambda\leqq C-\mu$ (and consequently $\mu\leqq C-\lambda$). Then if*

(26)
$$a\leqq\frac{C-1}{C-\mu}$$

and

(27)
$$b\leqq\frac{C-1}{C-\lambda}\,,$$

condition (10) *of Section* 3 *is satisfied, provided that*

(28)
$$\lambda\leqq T\leqq C-\mu$$

and consequently

(29)
$$\mu\leqq T^*\leqq C-\lambda\,.$$

PROOF. By Theorem 5 we have only to prove in the case where a or b is larger than 1. Let $a>1$, and for arbitrary i $(0\leqq i\leqq G-1)$ take $i'=(i+T-aT)^+ +T^*$. Then we have

$$\begin{aligned}{}_{T1}p_{ii'}&=P_r\{(i+x_n-aT)^+ +x_n'=(i+T-aT)^+ +T^*\}\\&\geqq P_r(x_n=T;\ x_n'=T^*)=p_Tp^*_{T^*}>0\,.\end{aligned}$$

But $i'>i$. In fact when $i\geqq T(a-1)$, using (26), (28) and (29) we get $i'=i+T-aT+T^*\geqq i-(a-1)\ (C-\mu)+\mu=(i+1)-a(C-\mu)+(C-1)\geqq(i+1)-(C-1)+(C-1)=i+1>i$. When $0\leqq i<T(a-1)$, $i'=T^*\geqq\mu$, but $i+1\leqq T(a-1)\leqq(C-\mu)(a-1)\leqq(C-1)-(C-\mu)=\mu-1<\mu$, hence $i'>i$ in this case, too. Thus the condition (10) holds for ${}_{T1}p_{ij}$. In the same way using (27), (28) and (29) we can conclude that the condition (10) is satisfied for ${}_{T2}p_{ij}$ when $b>1$.

6. A simple numerical example

Here we carried out a numerical computation for the case where $C=10$, $A=B=10$ and $a=0.7$, $b=0.5$, the arrivals of vehicles in S–N and W–E flows being Poisson processes respectively with $\alpha=0.8$ and $\beta=0.6$. Computation was carried out for four values of T (or T^*) given in Table 1.

For these values of T, m_0 in S–N flow and m_i $(0\leqq i\leqq5)$ in W–E flow were computed, and these values are shown in Table 2. Compu-

Table 1

Values of constants for given T (or T^*)

T	1.4	4.0	6.0	7.6
T^*	8.6	6.0	4.0	2.4
aT	1.	3.	4.	5.
αT	1.12	3.20	4.80	6.08
αT^*	6.88	4.80	3.20	1.92
bT^*	4.	3.	2.	1.
βT^*	5.16	3.60	2.40	1.44
βT	0.84	2.40	3.60	4.56
$\lambda(T)$	1.	2.	4.	5.

Remark: aT, bT^* and $\lambda(T)$ are respectively the nearest integers to their true values.

Table 2

Values of m_i for given T

T \ Flow / m	S-N	W-E					
	m_0	m_0	m_1	m_2	m_3	m_4	m_5
1.4	1.90	5.50	5.15	4.74	4.27	3.79	3.29
4.0	2.44	3.79	3.57	3.28	2.96	2.63	2.29
6.0	2.98	2.95	2.78	2.56	2.31	2.05	1.79
7.6	3.83	2.47	2.32	2.12	1.92	1.70	1.49

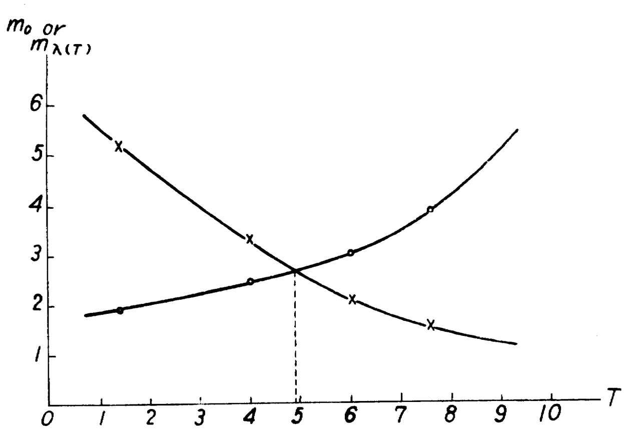

Fig. 5.

tation was easily carried out by means of FACOM 128, a digital computer installed in the Institute of Statistical Mathematics.

Using these values we roughly drew the graphs of functions $f(T)$

and $g(T)$ (cf. Fig. 5). From Fig. 5 it is seen that the optimum value of T is roughly equal to 4.9.

7. Summary and Acknowledgements

The present paper is an attempt to obtain any reasonable method of traffic control at a given intersection on the basis of probability theory, and as the first step a simple type of traffic was considered, under the control method by the repeated fixed-cycle traffic lights.

With regard to the efficiency of traffic control, a concept of optimum method of control was introduced from the point of view of balanced prolongation of the occurrence of uncontrollable confusion for two mutually intersecting flows of vehicles, and then the problem to determine this method of control was studied. For this purpose we considered two types of models concerning the arrivals of vehicles of both flows, and made the problem resolve into a certain general type of random walk.

Certain conditions were given under which we could solve our problem. Incidentally a certain type of matrix was studied, and the necessary and sufficient conditions for the matrix to be non-singular were given.

In some special cases we showed that one of the above-mentioned conditions was satisfied and our problem could be solved.

Finally the result of a numerical computation was given and we showed that our treatment was feasible in practice.

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