"ON THE CONVERGENCE OF PROJECTED DISTRIBUTIONS"*

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1. Introduction

Let X_n, Y_n $(n=1, 2, \cdots)$ be real random variables with "laws" or cumulative distribution functions (c.d.f.'s) $L(X_n)$, $L(Y_n)$ and characteristic functions (c.f.'s) $\varphi_{X_n}(t)$, $\varphi_{Y_n}(t)$ respectively. Suppose, as we shall throughout, that there are c.d.f.'s $F(x)$, $G(x)$ with corresponding c.f.'s $f(t)$, $g(t)$ such that

$$(1) \qquad L(X_n) \to F\,, \qquad L(Y_n) \to G\,,$$

i.e., X_n and Y_n converge in law or distribution. If $*$ denotes convolution, we seek conditions under which $L(cX_n+Y_n) \to F\left(\frac{x}{c}\right) * G(x)$, for some fixed positive c (mutatis mutandis for $c<0$) or equivalently for any fixed $c \neq 0$,

$$(2) \qquad \lim_{n\to\infty} \varphi_{cX_n+Y_n}(t) = f(ct)\, g(t)$$

for all real t. It is well known that the independence of X_n and Y_n for all n or the degeneracy[1] of F or G is ample to guarantee this for *all* real c. In fact, if X_n and Y_n are merely asymptotically independent, that is

$$\lim_{n\to\infty} E\, e^{i[t_1X_n+t_2Y_n]} = f(t_1)\, g(t_2)$$

then (2) is the immediate consequence of the substitution $t_1=ct$, $t_2=t$. As customary, E is the expected value operator.

Note that the distribution of cX_n+Y_n arises from the projection of the 2-dimensional mass distribution of X_n, Y_n on the ray through the origin with slope $1/c$.

REMARK. Clearly, (2) holds when $\varphi_{cX_n+Y_n}(t) = \varphi_{X_n}(ct)\,\varphi_{Y_n}(t)$, hence when X_n and Y_n are independent $(n=1, 2, \cdots)$. On the other hand, if

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1) A c.d.f. $F(x)$ is called degenerate if it has only one point of increase (with unit saltus).

$c=1$, $Y_n=X_n$ then (2) holds if and only if $f(2t)=f^2(t)$, a severe limitation. If, further, $\int_{-\infty}^{\infty} x^2\,dF(x)<\infty$, then F is degenerate.

For if X and Y are independent random variables, each having c.f. $f(t)$, the prior functional equation asserts that $2X$ and $X+Y$ are equidistributed; hence, from a result of Linnik (Dokl. Akad. Nauk SSSR 89 (1953) pp. 9–11), X is normally distributed, but then the functional equation coerces $f(t)$ to equal e^{iat} for some real a.

2. Convergence[2]

As the point of departure, we recall the unusual definition of the independence of two random variables proffered by Wintner. In [3, p. 5], X and Y are defined to be statistically independent if

$$L(X+Y)=L(X)*L(Y) \tag{3}$$

That this is not equivalent to the standard definition (in terms of product measure) is shown by an example of Cramér [see section 3]. Note, however, that if $EX^k<\infty$, $EY^k<\infty$ for some integer $k>1$, (3) implies

$$\sum_{j=0}^{k}\frac{(it)^j}{j!}E(X+Y)^j+o(t^k)\equiv\left[\sum_{j=0}^{k}\frac{(it)^j}{j!}EX^j+o(t^k)\right]\left[\sum_{j=0}^{k}\frac{(it)^j}{j!}EY^j+o(t^k)\right]$$

which in turn requires that

$$\sum_{j=1}^{r-1}\binom{r}{j}\operatorname{Cov}(X^j, Y^{r-j})=0\,, \qquad r=2, 3, \cdots, k$$

where $\operatorname{Cov}(X, Y)=E[X-EX][Y-EY]$. In particular ($k=2$), if it is meaningful to speak of the existence or lack of correlation between X and Y, i.e., $E[X-EX]^2$ and $E[Y-EY]^2$ are non-zero finite quantities, (3) implies that X and Y are uncorrelated.

Let I be the class of all infinitely differentiable c.f.'s; let I_1 be the subclass such that $\varphi\in I$, $\varphi_1\in I_1$, $D_k[\varphi(t)]_{t=0}=\varphi^{(k)}(0)=\varphi_1^{(k)}(0)$ ($k=1, 2, \cdots$) imply $\varphi(t)=\varphi_1(t)$. For instance, $\varphi\in I_1$ if $\varphi\in I$ and $\sum_{j=0}^{\infty}|\varphi^{(2j)}(0)|^{-1/2j}=\infty$ (Carleman). This is true in turn when $\varphi(t)$ is analytic at $t=0$, e.g., when the corresponding random variable has a finite range.

LEMMA 1. *Let* $\varphi(t)\in I_1$, $\varphi_n(t)\in I$ ($n=1, 2, \cdots$). *In order that*

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2) The writer cordially thanks his colleagues Dr. J. H. B. Kemperman and Dr. J. I. Rosenblatt for helpful suggestions.

(4) $$\lim_{n\to\infty}\varphi_n(t)=\varphi(t)\,, \qquad -\infty<t<\infty$$

it is sufficient that

(5) $$\lim_{n\to\infty}\varphi_n^{(k)}(0)=\varphi^{(k)}(0)$$

for all positive integers k. If, for infinitely many even k, the sequence $\{\varphi_n^{(k)}(0)\}$ *is bounded, then a necessary condition for* (4) *is that* (5) *holds for all* $k=1, 2, \cdots$.

PROOF. cf. [2, p. 184–185].

When it is unnecessary or inconvenient to emphasize the random variable in question, we shall denote $\varphi_{X_n}(t)$, $\varphi_{Y_n}(t)$, $\varphi_{cX_n+Y_n}(t)$ by $f_n(t)$, $g_n(t)$ and $h_n(t)$ respectively.

THEOREM 1. *Suppose that for each* $n=1, 2, \cdots$, $f_n(t)\in I$, $g_n(t)\in I$, $f(ct)g(t)\in I_1$. *Moreover, suppose that for each* $k=1, 2, \cdots$ *the sequences* $\{f_n^{(k)}(0)\}$ *and* $\{g_n^{(k)}(0)\}$ *are bounded. Then, in order that* (2) *holds, it is sufficient that*

(6) $$\lim_{n\to\infty}\sum_{j=1}^{r-1}\binom{r}{j}c^j\,\mathrm{Cov}\,(X_n^j,\, Y_n^{r-j})=0$$

for all positive integers r. Conversely, (2) *implies that* (6) *holds for every integer* $r\geqq 2$.

PROOF. For each k, $\{D_k[f_n(ct)\cdot g_n(t)]_{t=0}\}$ is a bounded sequence. Further, from (1), $\lim_{n\to\infty} f_n(ct)g_n(t)=f(ct)g(t)$.

Hence, from lemma 1, for each k

(7) $$\lim_{n\to\infty} D_k[f_n(ct)\,g_n(t)]_{t=0}=D_k[f(ct)\,g(t)]_{t=0}\,.$$

Also, the sequence $\{h_n^{(k)}(0)\}=\{i^kE(cX_n+Y_n)^k\}$ is easily seen to be bounded $(k=1, 2, \cdots)$. Again, using lemma 1, it follows that (2) holds if

(8) $$\lim_{n\to\infty} h_n^{(k)}(0)=D_k[f(ct)\,g(t)]_{t=0}\,.$$

for all positive k and only if (8) holds for all k. The assertion of the theorem now follows from (7), (8) and

(9) $$h_n^{(k)}(0)-D_k[f_n(ct)\,g_n(t)]_{t=0}=i^k\sum_{j=1}^{k-1}\binom{k}{j}c^j\,\mathrm{Cov}\,(X_n^j,\, Y_n^{k-j})\,.$$

THEOREM 2. *Let* X, Y *be real random variables with joint cf.* $\varphi(z_1, z_2)$. *Suppose that* $\psi_1(z)=\varphi(z, 0)$ *and* $\psi_2(z)=\varphi(0, z)$ *are analytic for* $|z|\leqq r_1$, $|z|\leqq r_2$, *respectively. Then* $\varphi(z_1, z_2)$ *is analytic for* $|\vartheta(z_1)|<1/2\, r_1$, $|\vartheta(z_2)|<1/2\, r_2$.

PROOF. Since $\sum_{k=0}^{\infty}|\psi_j^{(k)}(0)|r_j^k/k!$ is convergent, we have $|\psi_j^{(k)}(0)|\leqq C_j k!\, r_j^{-k}$ for $j=1, 2$ where C_1, C_2 are absolute constants. Further,

$$\begin{aligned}|\varphi^{(k,m)}(0, 0)| &\leqq E|X|^k|Y|^m \leqq \sqrt{E|X|^{2k}E|Y|^{2m}}\\ &\leqq C_1C_2r_1^{-k}r_2^{-m}\sqrt{(2k)!\,(2m)!}\\ &\leqq C(1/2\, r_1)^{-k}(1/2\, r_2)^{-m}\, k!\, m!\end{aligned}$$

since $\lim_{n\to\infty}\frac{(2n)!}{(2^n n!)^2}=0$. The assertion now follows from a straight-forward generalization of a standard procedure (cf. [1, p. 177], [2, p. 212]).

REMARK. Taking $X\equiv Y$, we have $\varphi(t_1, t_2)=\psi_1(t_1+t_2)$ showing the stated region of analyticity cannot be improved very much.

COROLLARY. *Under the hypothesis of Theorem* 2, $\varphi(z_1, z_2)$ *defines an analytic function for* $|\vartheta(z_1)|<\theta\, r_1$, $|\vartheta(z_2)|<(1-\theta)r_2$ *for any* θ *in* $(0, 1)$.

PROOF. Now $Ee^{r_1|X|}<\infty$, $Ee^{r_2|Y|}<\infty$ (cf. [2, p. 212]). Hence, by Holder's inequality, $Ee^{\theta r_1|X|+(1-\theta)r_2|Y|}<\infty$. Thus, the integral

$$E[\exp\{iz_1X+iz_2Y\}]=\varphi(z_1, z_2)$$

converges and defines an analytic function in $|\vartheta(z_1)|<\theta\, r_1$, $|\vartheta(z_2)|<(1-\theta)\, r_2$. The optimum choice seems $\theta=\frac{r_2}{r_1+r_2}$.

We are now in a position to prove:

THEOREM 3. *Suppose that* $f_n(z)$, $g_n(z)$ *are analytic in* $|z|\leqq r_1$, $|z|\leqq r_2$ *respectively and that* $f(ct)\ g(t)$ *has a unique extension from an* $\frac{r_1r_2}{|c|r_1+r_2}\sqrt{1+c^2}$ *neighborhood of the origin. Let*

$$a_{n,k}(c)=\sum_{j=1}^{k-1}\frac{c^j}{j!\,(k-j)!}\operatorname{Cov}(X_n^j, Y_n^{k-j})$$

and suppose further that for some positive number $\varepsilon>\frac{r_1r_2\sqrt{1+c^2}}{|c|r_1+r_2}$,

$$\lim_{n\to\infty} \sum_{1}^{\infty} |a_{nk}(c)|\varepsilon^k = 0 \ .$$

Then (2) *holds.*

PROOF. From [2, p. 211], it suffices to prove that (2) holds in a neighborhood of zero. Then, according to (1), it suffices to demonstrate that

$$\lim_{n\to\infty} [h_n(t) - f_n(ct)\, g_n(t)] = 0$$

holds in a neighborhood of the origin. However, from the corollary to Theorem 2 $h_n(t)$ is analytic (Take $(1-\theta)r_2 = |c|\theta r_1$) for $|t| < \frac{r_1 r_2}{|c|r_1 + r_2}\sqrt{1+c^2} = \delta$ (say). From (9), for $|t|<\delta$

$$h_n(t) - f_n(ct)\, g_n(t) = \sum_{1}^{\infty} a_{nk}(it)^k \ ; \tag{10}$$

hence, the left hand side of (10) tends to zero for $|t|<\delta$.

If, in Theorem 1, we take $X_n = X$, $Y_n = Y$ $(n=1, 2, \cdots)$, we obtain the following:

COROLLARY. *If* $f(t), g(t) \in I$ *and* $f(ct)g(t) \in I$, *then* $cX+Y$ *has the characteristic function* $f(ct)g(t)$ *if*

$$0 = \sum_{j=1}^{r-1} \binom{r}{j} c^j \operatorname{Cov}(X^j, Y^{r-j}) = P_r(c) \text{ (say)} \tag{11}$$

for all positive integers r *and only if* (11) *holds for all integers* $r \geqq 2$ (*in particular,* X *and* Y *are uncorrelated*).

Now, (11) will certainly hold if

$$\operatorname{Cov}(X^j, Y^k) = 0 \qquad \text{for all } j, k = 1, 2, \cdots \ . \tag{12}$$

Suppose this to be the case and in addition that $f(t)$ and $g(t)$ are analytic, say for $|t|<2r_1$ and $|t|<2r_2$ respectively. Then (11) is valid for all integers $r \geqq 2$ and all real c, whence by the corollary just cited, $\varphi_{cX+Y}(t) = \varphi_X(ct)\varphi_Y(t)$, for all real c. But then if (θ, t) is the polar coordinate representation of a point (t_1, t_2) of the real plane and $c = \cot\theta$, $\theta \neq 0$,

$$\begin{aligned}\varphi_{X,Y}(t_1, t_2) &= \varphi_{X\cos\theta,\, Y\sin\theta}(t, t) \\ &= \varphi_{X\cos\theta + Y\sin\theta}(t) = \varphi_{cX+Y}(t\sin\theta) \\ &= \varphi_{cX}(t\sin\theta)\,\varphi_Y(t\sin\theta) = \varphi_X(t_1)\,\varphi_Y(t_2) \ .\end{aligned}$$

Consequently, X and Y are independent in this case. An alternative argument, due to J. H. B. Kemperman, consists in noting that by Theorem 2

$$\varphi(t_1, t_2)=\sum_{j,k} E(X^j Y^k)\frac{(it_1)^j (it_2)^k}{j!\,k!}$$

is analytic in a neighborhood of (0, 0), that according to (12), $EX^jY^k = EX^j \cdot EY^k$ and hence $\varphi(t_1, t_2)=f(t_1)g(t_2)$, first, in a neighborhood of the origin and then for all real (t_1, t_2) by analytic continuation.

If (11) obtains and (12) does not, let r be the smallest positive integer with $\mathrm{Cov}\,(X^j, Y^{r-j})\neq 0$ for some (and therefore at least two such) $j=1, 2, \cdots, r-1$. Again, by the corollary, since $P_r(c)$ has constant term zero,

$$\varphi_{cX+Y}(t)=\varphi_X(ct)\cdot\varphi_Y(t)\ , \qquad \text{all real } t \tag{13}$$

for at most $r-2$ non-zero values of c. We have thus proved:

THEOREM 4. *If X and Y have analytic characteristic functions and* $\mathrm{Cov}\,(X^j, Y^k)=0$ *for all integers $j, k\geqq 1$, then X and Y are independent random variables. Conversely, if $r\geqq 2$ is the smallest integer with* $\mathrm{Cov}\,(X^j, Y^{r-j})\neq 0$ *for some $j=1, 2, \cdots, r-1$, then* (13) *obtains for at most $r-2$ non-zero values of c.*

3. Extension and Examples

Under the hypothesis of Theorem 1, we have even $\lim_{n\to\infty}\varphi_{c_nX_n+Y_n}(t)= f(ct)g(t)$ where c_n is a sequence of real constants approaching c.

If in (6), X_n and Y_n are replaced by $\log X_n$ and $\log Y_n$ (where X_n and Y_n are now presumed to be positive r.v.'s), we obtain a condition for $L(X_n^{cb}\cdot Y_n^b)\to H$ where $b\neq 0$ is an arbitrary constant and

$$\begin{aligned} H(z)&=\int_{x^{cb}}\int_{y^b<z} dF(x)\,dG(y)\ , \qquad z>0 \\ &=0\ , \qquad z\leqq 0\ . \end{aligned}$$

Results analogous to those of section 2 may be obtained for the case of a sequence of k dimensional vectors $X_n=(X_{n1}, X_{n2}, \cdots, X_{nk})$ where it is known that $L(X_{nj})\to F_j$, $j=1, 2, \cdots, k$ and it is desired that

$$L\Big(\sum_{j=1}^{k} c_j X_{nj}\Big) \to F_1\Big(\frac{x}{c_1}\Big) * F_2\Big(\frac{x}{c_2}\Big) * \cdots * F_k\Big(\frac{x}{c_k}\Big) .$$

In the following example of Cramér ($\theta=1$) already alluded to, $r=4$ while $X+Y$ and $-X+Y$ are precisely the two linear combinations with the property (13):

$$f_\theta(x, y)=\frac{1}{4\theta^2}[1+xy(x^2-y^2)] , \quad \text{if} \quad |x|\leqq\theta , \ |y|\leqq\theta$$

$$=0 , \qquad \text{otherwise.}$$

Here, both X and Y are uniformly distributed in $[-\theta, \theta]$. Theorem 1 may be trivially exemplified for $c=1$ or -1 by assigning X_n and Y_n the joint density function $f_{\theta_n}(x, y)$ where $\lim_{n\to\infty} \theta_n=\theta>0$.

The question naturally poses itself whether for every integer $r>2$ and arbitrary constants $c_1, c_2, \cdots, c_s$ ($s=r-2$), there exists a joint distribution such that (13) obtains for these and only these values of c (excluding the trivial value zero). The answer is affirmative.

Let $q(x)=\sin x, \quad -\pi\leqq x\leqq\pi$

$$0 , \qquad |x|>\pi$$

Its Fourier transform, say $Q(t)$ is

$$Q(t)=i\left[\frac{\sin \pi(1-t)}{1-t}-\frac{\sin \pi(1+t)}{1+t}\right]$$

an entire function of exponential order 1 with $Q(0)=0$. Define

$$\Gamma(t_1, t_2)=Q(t_1)Q(t_2)\prod_{i=1}^{s} Q(t_1-c_it_2)$$

where $c_1, c_2, \cdots, c_s$ are the previously specified constants. Then

$$\Gamma(t, 0)=\Gamma(0, t)=\Gamma(c_it, t)=0 \qquad \text{for all real } t, \quad i=1, 2, \cdots, s$$

and since $\Gamma(t_1, t_2)$ is of exponential order, entire, its inverse Fourier transform, say $\gamma(x, y)$ vanishes outside some bounded region A of the x, y plane. [cf. 4, p. 13] Finally, since $\gamma(x, y)$ is the convolution of functions one of which is continuous, it is bounded in A. Thus, if $f_1(x)$, $f_2(x)$ are probability density functions with spectrum $(-\infty, \infty)$ e.g., $f_1(x)=f_2(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, then for sufficiently small ε,

$$f(x, y)=f_1(x)f_2(y)+\varepsilon\gamma(x, y)$$

is a probability density function whose c.f.

$$\varphi(t_1, t_2) = \varphi_1(t_1)\varphi_2(t_2) + \varepsilon\Gamma(t_1, t_2)$$

where $\varphi_j(t) = \int_{-\infty}^{\infty} e^{itx} f_j(x)\, dx$, Clearly,

$$\varphi(c_j t, t) = \varphi_1(c_j t)\varphi_2(t) \qquad \text{for all real } t$$

and $j = 1, 2, \cdots, s$, and no other non-zero values of c.

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