ON SCHEDULING OF OVERTIME WORK

By YUKIO SUZUKI

(Received May 10, 1957)

1. Introduction

In this paper we treat the problem of scheduling of overtime work in a manufacturing enterprise which makes a kind of meter.

Suppose that this enterprise receives the order for the meter on a fixed day, say 25th, in every month and the custmer is only one monopolistic enterprise. Further, suppose that the day of supply for the order is specified, say 13th in every month. Of course, the amount of the order in a month is not known until it is received from the custmer on 25th. However, it is known that the amount of the order in every month is not less than that of meters which can be produced by the standard work. Therefore the excess amount of the order must be produced by overtime work.

In our enterprise there are three kinds of overtime work:

the 1st kind: from 4.30 *p.m.* to 7.30 *p.m.* Overtime pay is increased by 25% of the standard pay

the 2nd kind: from 7.30 *p.m.* to 10.00 *p.m.* Overtime pay is increased by 30% of the standard pay

the 3rd kind: from 10.00 *p.m.* to the following morning. Overtime pay is increased by 50% of the standard pay

Here, the problem arises about what amount of overtime work during the period in which the order is not received, say 13th through 25th, is to be made. If we do no overtime work, only standard work in this period, and if the order is so big that we cannot meet it without a large amount of the 3rd kind of overtime work in the succeeding period, say 26th—12th, the amount to pay for the overtime work will be considerably great. On the contrary, if we do some overtime work in the preceeding period and thus decrease the amount of overtime work in the succeeding period, we can do with less amount to pay for the overtime work. Of course, in this problem it is essentially important to obtain as much information as possible to predict the amount of the coming order. Using all available information we shall be able to set up the a priori distribution of the coming order. On the basis of this distribution we can determine a reasonable scheduling of overtime work.

2. Formulation of the problem and the solution

Suppose that, in the capacity of the enterprise, the total amount of meters produced by standard work in a month, is N and the amounts produced per hour by the 1st kind of overtime work, the 2nd kind of overtime work and the 3rd kind of overtime work are n_1, n_2 and n_3, respectively. Further, let the average wage per hour of standard work, the 1st kind of overtime work, the 2nd kind of overtime work and the 3rd kind of overtime work be c, c_1, c_2 and c_3. It seems natural to assume that the following inequalities hold :

$$(1) \qquad n_1 > n_2 > n_3\,, \qquad c_1 < c_2 < c_3$$

Let T_1, T_2 and T_3 be the possible amount of time in a month for the 1st, 2nd and 3rd kinds of overtime work, respectively. Let the ratio of the lengths of the two periods, that is, the period where the coming order is not known and the succeeding period where the order is specified, be $\alpha:\beta$, where $\alpha+\beta=1$. Our scheduling of overtime work is to decide the amount of time, x, for production by overtime work during the period in which the order is not received.

Let the comming order be a random variable W with a discrete distribution function $G_0(w)$. But, for the convenience of the later treatment, we replace $G_0(w)$ by $G(w)$ which is continuous, has a density function and is approximately close to $G_0(w)$.

As described in section 1, we assume

$$(2) \qquad Y=W-N\geqq 0$$

If we produce more than the order during the period where the order is not received, a penalty is imposed on the excess products. Assume this penalty is p yen per excess product.

Now, let the corresponding expense be $C(x, Y)$ for the policy x. Then the expected expense for the policy x is

$$E[C(x, Y)]=\int_0^\infty C(x, y)dF(y)=\int_0^\infty C(x, y)f(y)dy\,,$$

where

$$F(y)=G(y+N)\,,\quad f(y)=\frac{dF(y)}{dy}\,.$$

A policy $\tilde{x}$ is called optimal if

$$E[C(\tilde{x},\ Y)]=\min_{x} E[C(x,\ Y)]\,.$$

To specify the functional form of $C(x, y)$, we treat the problem separately in the following three cases:

(A) $$\alpha T_1 \geqq x \geqq 0\,,$$

(B) $$\alpha T_2+\alpha T_1 \geqq x \geqq \alpha T_1\,,$$

(C) $$\alpha T_3+\alpha T_2+\alpha T_1 \geqq x \geqq \alpha T_2+\alpha T_1\,.$$

From the practical point of view, we assume here

(3) $$F(\beta T_1+\beta T_2+\beta T_3)=1$$

This means that, without use of any overtime work in the first period, we can supply products for the custmer. Therefore, under this restriction we need not treat the case (C).

Case (A)

(i) if $n_1 x \leqq y \leqq n_1 x + n_1 T_1 \beta$,

$$C(x,\ y)=\frac{c_1 y}{n_1} \equiv C_{11}(x,\ y)$$

(ii) if $n_1 x+n_1 T_1 \beta \leqq y \leqq n_1 x + n_1 T_1 \beta + n_2 T_2 \beta$,

(4) $$C(x,\ y)=c_1(x+\beta T_1)+\frac{c_2}{n_2}(y-n_1 x-n_1 \beta T_1) \equiv C_{12}(x,\ y)$$

(iii) if $n_1 x+n_1 T_1 \beta+n_2 T_2 \beta \leqq y \leqq n_1 x + n_1 T_1 \beta + n_2 T_2 \beta + n_3 T_3 \beta$

$$C(x,\ y)=c_1(x+\beta T_1)+c_2 T_2 \beta+\frac{c_3}{n_3}(y-n_1 x-n_1 \beta T_1-n_2 \beta T_2) \equiv C_{13}(x,\ y)$$

(iv) if $0 \leqq y \leqq n_1 x$

$$C(x,\ y)=c_1 x+p(n_1 x-y) \equiv C_{14}(x,\ y)$$

Hence the expected expense of the case (A) is

$$\underset{(A)}{E}[C(x,\ Y)]=\int_0^{n_1 x} C_{14}(x,\ y)f(y)dy+\int_{n_1 x}^{n_1 x+n_1 T_1 \beta} C_{11}(x,\ y)f(y)dy$$

$$\int_{n_1 x+n_1 T_1 \beta}^{n_1 x+n_1 T_1 \beta+n_2 T_2 \beta} C_{12}(x,\ y)f(y)dy+\int_{n_1 x+n_1 T_1 \beta+n_2 T_2 \beta}^{n_1 x+n_1 T_1 \beta+n_2 T_2 \beta+n_3 T_3 \beta} C_{13}(x,\ y)f(y)dy$$

and its first derivative is

$$\frac{d\underset{(A)}{E}[C(x,\ Y)]}{dx}=n_1C_{14}(x,\ n_1x)f(n_1x)+\int_0^{n_1x}\frac{\partial C_{14}}{\partial x}f(y)dy$$

$$+n_1C_{11}(x,\ n_1x+n_1T_1\beta)f(n_1x+n_1T_1\beta)-n_1C_{11}(x,\ n_1x)f(n_1x)$$

$$+\int_{n_1x}^{n_1x+n_1T_1\beta}\frac{\partial C_{11}}{\partial x}f(y)dy$$

$$+n_1C_{12}(x,\ n_1x+n_1T_1\beta+n_2T_2\beta)f(n_1x+n_1T_1\beta+n_2T_2\beta)$$

$$-n_1C_{12}(x,\ n_1x+n_1T_1\beta)f(n_1x+n_1T_1\beta)$$

$$+\int_{n_1x+n_1T_1\beta}^{n_1x+n_1T_1\beta+n_2T_2\beta}\frac{\partial C_{12}}{\partial x}f(y)dy$$

$$+n_1C_{13}(x,\ n_1x+n_1T_1\beta+n_2T_2\beta+n_3T_3\beta)$$

$$\times f(n_1x+n_1T_1\beta+n_2T_2\beta+n_3T_3\beta)$$

$$-n_1C_{13}(x,\ n_1x+n_1T_1\beta+n_2T_2\beta)f(n_1x+n_1T_1\beta+n_2T_2\beta).$$

From (3) and (4) we have

$$f(n_1x+n_1T_1\beta+n_2T_2\beta+n_3T_3\beta)=0$$

$$\frac{\partial C_{11}}{\partial x}=0\,,\quad \frac{\partial C_{12}}{\partial x}=c_1-\frac{c_2n_1}{n_2}\,,\quad \frac{\partial C_{13}}{\partial x}=c_1-\frac{c_3n_1}{n_3}\,,\quad \frac{\partial C_{14}}{\partial x}=c_1+pn_1$$

$$C_{14}(x,\ n_1x)=C_{11}(x,\ n_1x)\,,\quad C_{11}(x,\ n_1x+n_1T_1\beta)=C_{12}(x,\ n_1x+n_1T_1\beta)$$

$$C_{12}(n,\ n_1x+n_1T_1\beta+n_2T_2\beta)=C_{13}(n,\ n_1x+n_1T_1\beta+n_2T_2\beta)\,.$$

Therefore we obtain

$$\frac{d\underset{(A)}{E}[C(x,\ Y)]}{dx}=\left(c_1-\frac{c_2n_1}{n_2}\right)[F(n_1x+n_1T_1\beta+n_2T_2\beta)-F(n_1x+n_1T_1\beta)]$$

$$+\left(c_1-\frac{c_3n_1}{n_3}\right)[1-F(n_1x+{}_1Tn_1\beta+n_2T_2\beta)]$$

$$+(c_1+pn_1)[F(n_1x)-F(0)]$$

$$\frac{d^2\underset{(A)}{E}[C(x,\ Y)]}{dx^2}=n_1\left(\frac{c_3n_1}{n_3}-\frac{c_2n_1}{n_2}\right)f(n_1x+n_1T_1\beta+n_2T_2\beta)$$

$$+n_1\left(pn_1+\frac{c_2n_1}{n_2}\right)f(n_1x+n_1T_1\beta)+n_1(c_1+pn_1)f(n_1x)$$

Taking into account of (1), we have

$$\frac{d^2\underset{(A)}{E}[C(x,\ Y)]}{dx^2}\geqq 0\,.$$

Let the solution of the following equation be x_1.

$$(5)\quad \left(c_1-\frac{c_2n_1}{n_2}\right)[F(n_1x+n_1T_1\beta+n_2T_2\beta)-F(n_1x+n_1T_1\beta)]$$
$$+\left(c_1-\frac{c_3n_1}{n_3}\right)[1-F(n_1x+n_1T_1\beta+n_2T_2\beta)]+(c_1+pn_1)F(n_1x)=0.$$

Then $\underset{(A)}{E}[C(x, Y)]$ takes its minimum value for the value $\tilde{x}_A$, such that

$$\tilde{x}_A=\begin{cases} x_1 & \text{if } \ 0\leqq x_1\leqq \alpha T_1 \\ \alpha T_1 & \text{if } \ \alpha T_1 < x_1 \end{cases}$$

Case (B)

(i) if $n_1\alpha T_1+n_2(x-\alpha T_1)\leqq y\leqq n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta$

$$C(x, y)=\frac{c_1}{n_1}y+\left(c_2-\frac{c_1n_2}{n_1}\right)(x-\alpha T_1)\equiv C_{21}(x, y)$$

(ii) if $n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta\leqq y\leqq n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta$,

$$C(x, y)=\frac{c_2}{n_2}y+c_1T_1-c_2\left(\frac{n_1}{n_2}T_1+\beta T_2\right)\equiv C_{22}(x, y)$$

(iii) if $n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta\leqq y\leqq n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta+n_3T_3\beta$,

$$C(x, y)=\frac{c_3}{n_3}y+\left(c_2-\frac{c_3n_2}{n_3}\right)x-\left(c_2-\frac{c_3n_2}{n_3}\right)\alpha T_1+\left(c_1-\frac{c_3n_1}{n_3}\right)T_1$$
$$+\left(c_2-\frac{c_3n_2}{n_3}\right)T_2\beta$$
$$\equiv C_{23}(x, y)$$

(iv) if $0\leqq y\leqq n_1\alpha T_1+n_2(x-\alpha T_1)$

$$C(x, y)=c_1\alpha T_1+c_2(x-\alpha T_1)+p\{n\alpha T_1+n_2(x-\alpha T_1)-y\}$$
$$\equiv C_{24}(x, y)$$

Hence the expected expense of the case (B) is

$$\underset{(B)}{E}[C(x, Y)]=\int_0^{n_1\alpha T_1+n_2(x-\alpha T_1)}C_{24}(x, y)f(y)dy+\int_{n_1\alpha T_1+n_2(x-\alpha T_1)}^{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta}C_{21}(x, y)f(y)dy$$
$$+\int_{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta}^{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta}C_{22}(x, y)f(y)dy$$
$$+\int_{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta}^{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta+n_3T_3\beta}C_{23}(x, y)f(y)dy$$

In the same way as in case (A), we have

$$(6)\quad \frac{d\underset{(B)}{E}[C(x,\ Y)]}{dx}=(c_2+pn_2)[F\{n_1\alpha T_1+n_2(x-\alpha T_1)\}-F(0)\}$$

$$+\left(c_2-\frac{c_1n_2}{n_1}\right)[F\{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta\}$$

$$-F\{n_1\alpha T_1+n_2(x-\alpha T_1)\}]$$

$$+\left(c_2-\frac{c_3n_2}{n_3}\right)[1-F\{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta\}]$$

$$\frac{d^2\underset{(B)}{E}[C(x,\ Y)]}{dx^2}=pn_2^2f\{n_1\alpha T_1+n_2(x-\alpha T_1)\}$$

$$+n_2\left(c_2-\frac{c_1n_2}{n_1}\right)f\{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta\}$$

$$+\frac{c_1n_2^2}{n_1}f\{n_1\alpha T_1+n_2(x-\alpha T_1)\}$$

$$+n_2\left(\frac{c_3n_2}{n_3}-c_2\right)f\{n_1\alpha T_1+n_2(x-\alpha T_1)+n_1T_1\beta+n_2T_2\beta\}$$

$$\geqq 0\,.$$

Let the solution of the following equation be x_2.

$$(7)\quad \frac{d\underset{(B)}{E}[\mathrm{C}(x,\ Y)]}{dx}=0\,.$$

Then $\underset{(B)}{E}[C(x,\ Y)]$ takes its minimum value for the value $\tilde{x}_B$, such that

$$\tilde{x}_B=\begin{cases} x_2 & \text{if} \quad \alpha T_1\leqq x_2\leqq \alpha T_1+\alpha T_2 \\ \alpha T_1 & \text{if} \quad x_2<\alpha T_1 \\ \alpha T_1+\alpha T_2 & \text{if} \quad \alpha T_1+\alpha T_2<x_2 \end{cases}$$

Thus a good policy is determined as follows:

$$\text{when } \min\{\underset{(A)}{E}[C(\tilde{x}_A,\ Y)],\ \underset{(B)}{E}[C(\tilde{x}_B,\ Y)]\}=\begin{cases}\underset{(A)}{E}, \text{ then take } \tilde{x}_A \\ \underset{(B)}{E}, \text{ then take } \tilde{x}_B\end{cases}$$

THE INSTITUTE OF STATISTICAL MATHEMATICS