ON THE ERRORS OF OUTPUTS DUE TO ERRORS OF TECHNICAL COEFFICIENTS IN LEONTIEF'S OPEN INPUT-OUTPUT MODELS

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1. In the Leontief's open input-output models, technical coefficients are obtained from statistical data, so they cannot always be kept from errors. The amount of final demands, however, are considered as the independent variables, which are assumed in this article to be given without errors exogenously. The purpose of this article is to evaluate the errors of dependent variables, that is, those of the outputs of industries caused by errors of the technical coefficients.

2. Let Y_i denote the amount of the final demand in the i-th industry. Each technical coefficient can be written as

$$a_{ij}+\varepsilon_{ij} \tag{1}$$

where a_{ij} is the true coefficient and ε_{ij} is a random error. Let X_i be the true output of the i-th industry. Then X_i $(i=1, 2, \cdots, n)$ satisfy the following matric equation :

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}. \tag{2}$$

For the simplicity of notation, we write this equation as

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}, \tag{3}$$

or

$$BX=Y. \tag{4}$$

Now, let X'_i be the value of the output of the i-th industry. Then X'_i $(i=1, 2,\cdots, n)$ satisfy the following matric equation :

$$(5)\qquad \begin{pmatrix} 1-a_{11}-\varepsilon_{11} & -a_{12}-\varepsilon_{12} & \cdots & -a_{1n}-\varepsilon_{1n} \\ -a_{21}-\varepsilon_{21} & 1-a_{22}-\varepsilon_{22} & \cdots & -a_{2n}-\varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1}-\varepsilon_{n1} & -a_{n2}-\varepsilon_{n2} & \cdots & -a_{nn}-\varepsilon_{nn} \end{pmatrix} \begin{pmatrix} X_1' \\ X_2' \\ \cdots \\ X_n' \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}.$$

This equation can be written as

$$(6)\qquad \begin{pmatrix} b_{11}-\varepsilon_{11} & b_{12}-\varepsilon_{12} & \cdots & b_{1n}-\varepsilon_{1n} \\ b_{21}-\varepsilon_{21} & b_{22}-\varepsilon_{22} & \cdots & b_{2n}-\varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}-\varepsilon_{n1} & b_{n2}-\varepsilon_{n2} & \cdots & b_{nn}-\varepsilon_{nn} \end{pmatrix} \begin{pmatrix} X_1' \\ X_2' \\ \cdots \\ X_n' \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix},$$

or

$$(7)\qquad B'X'=Y.$$

At first, we evaluate $|B'|$:

$$|B'| = \begin{vmatrix} b_{11}-\varepsilon_{11} & b_{12}-\varepsilon_{12} & \cdots & b_{1n}-\varepsilon_{1n} \\ b_{21}-\varepsilon_{21} & b_{22}-\varepsilon_{22} & \cdots & b_{2n}-\varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}-\varepsilon_{n1} & b_{n2}-\varepsilon_{n2} & \cdots & b_{nn}-\varepsilon_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} + \begin{vmatrix} -\varepsilon_{11} & b_{12} & \cdots & b_{1n} \\ -\varepsilon_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\varepsilon_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} b_{11} & -\varepsilon_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & -\varepsilon_{22} & b_{23} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & -\varepsilon_{n2} & b_{n3} & \cdots & b_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} b_{11} & \cdots & b_{1,n-1} & -\varepsilon_{1n} \\ b_{21} & \cdots & b_{2,n-1} & -\varepsilon_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{n,n-1} & -\varepsilon_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} -\varepsilon_{11} & -\varepsilon_{12} & b_{13} & \cdots & b_{1n} \\ -\varepsilon_{21} & -\varepsilon_{22} & b_{23} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\varepsilon_{n1} & -\varepsilon_{n2} & b_{n3} & \cdots & b_{nn} \end{vmatrix} + \cdots$$

and since a_{ij} is of order $1/n$, on neglecting the higher order terms of errors,

$$=|B| - \sum_{i,j=1}^{n} \varepsilon_{ij} b^{ij},$$

where b^{ij} is the cofactor of b_{ij} in $|B|$.

Now, put

$$B^{-1}=C=\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}.$$

Then this can be rewritten as

$$(8) \qquad |B'|=|B|\left(1-\sum_{i,j=1}^{n}\varepsilon_{ij}\frac{b^{ij}}{|B|}\right)=|B|\left(1-\sum_{i,j=1}^{n}\varepsilon_{ij}c_{ji}\right).$$

In the following, we evaluate b'^{ij}, the cofactor of $b_{ij}-\varepsilon_{ij}$ in $|B'|$.

$$b'^{ij}=(-1)^{i+j}\begin{vmatrix} b_{11}-\varepsilon_{11} & \cdots & b_{1,j-1}-\varepsilon_{1,j-1} & b_{1,j+1}-\varepsilon_{1,j+1} & \cdots & b_{1n}-\varepsilon_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i-1,1}-\varepsilon_{i-1,1} & \cdot & b_{i-1,j-1}-\varepsilon_{i-1,j-1} & b_{i-1,j+1}-\varepsilon_{i-1,j+1} & \cdot & b_{i-1,n}-\varepsilon_{i-1,n} \\ b_{i+1,1}-\varepsilon_{i+1,1} & \cdot & b_{i+1,j-1}-\varepsilon_{i+1,j-1} & b_{i+1,j+1}-\varepsilon_{i+1,j+1} & \cdot & b_{i+1,n}-\varepsilon_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1}-\varepsilon_{n1} & \cdots & b_{n,j-1}-\varepsilon_{n,j-1} & b_{n,j+1}-\varepsilon_{n,j+1} & \cdot & b_{nn}-\varepsilon_{nn} \end{vmatrix}$$

and neglecting the higher order terms of errors similarly as in the evaluation of $|B'|$,

$$=(-1)^{i+j}\begin{vmatrix} b_{11} & \cdots & b_{1,j-1} & b_{1,j+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i-1,1} & \cdots & b_{i-1,j-1} & b_{i-1,j+1} & \cdots & b_{i-1,n} \\ b_{i+1,1} & \cdots & b_{i+1,j-1} & b_{i+1,j+1} & \cdots & b_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{n,j-1} & b_{n,j+1} & \cdots & b_{nn} \end{vmatrix}$$

$$+(-1)^{i+j}\begin{vmatrix} -\varepsilon_{11} & \cdots & b_{1,j-1} & b_{1,j+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\varepsilon_{i-1,1} & \cdots & b_{i-1,j-1} & b_{i-1,j+1} & \cdots & b_{i-1,n} \\ -\varepsilon_{i+1,1} & \cdots & b_{i+1,j-1} & b_{i+1,j+1} & \cdots & b_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\varepsilon_{n1} & \cdots & b_{n,j-1} & b_{n,j+1} & \cdots & b_{nn} \end{vmatrix}$$

$$+\cdots+(-1)^{i+j}\begin{vmatrix} b_{11} & \cdots & b_{1,j-1} & b_{1,j+1} & \cdots & b_{1,n-1} & -\varepsilon_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i-1,1} & \cdots & b_{i-1,j-1} & b_{i-1,j+1} & \cdots & b_{i-1,n-1} & -\varepsilon_{i-1,n} \\ b_{i+1,1} & \cdots & b_{i+1,j-1} & b_{i+1,j+1} & \cdots & b_{i+1,n-1} & -\varepsilon_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n,1} & \cdots & b_{n,j-1} & b_{n,j+1} & \cdots & b_{n,n-1} & -\varepsilon_{nn} \end{vmatrix}$$

$$=b^{ij}-\sum_{\substack{k\neq i\\ l\neq j}}^{n}\varepsilon_{kl}b^{ij;kl}$$

where $b^{ij;kl}$ denotes $(-1)^{i+j}$ (the cofactor of b_{kl} in the minor of b^{ij}), that is, for example

$$(-1)^{i+j+k-1+l}\begin{vmatrix} b_{11} & \cdots & b_{1,l-1} & b_{1,l+1} & \cdots & b_{1,j-1} & b_{1,j+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i-1,1} & \cdots & b_{i-1,l-1} & b_{i-1,l+1} & \cdots & b_{i-1,j-1} & b_{i-1,j+1} & \cdots & b_{i-1,n} \\ b_{i+1,1} & \cdots & b_{i+1,l-1} & b_{i+1,l+1} & \cdots & b_{i+1,j-1} & b_{i+1,j+1} & \cdots & b_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k-1,1} & \cdots & b_{k-1,l-1} & b_{k-1,l+1} & \cdots & b_{k-1,j-1} & b_{k-1,j+1} & \cdots & b_{k-1,n} \\ b_{k+1,1} & \cdots & b_{k+1,l-1} & b_{k+1,l+1} & \cdots & b_{k+1,j-1} & b_{k+1,j+1} & \cdots & b_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{n,l-1} & b_{n,l+1} & \cdots & b_{n,j-1} & b_{n,j+1} & \cdots & b_{nn} \end{vmatrix}$$

and this can be written as $b^{kl;ij}$. Then we have

$$b'^{ij}=b^{ij}-\sum_{\substack{k\neq i\\ l\neq j}}^{n}\varepsilon_{kl}b^{kl;ij}\,. \tag{9}$$

Now,

$$X'=(B')^{-1}Y$$

$$=\begin{pmatrix} \frac{b'^{11}}{|B'|} & \frac{b'^{21}}{|B'|} & \cdots & \frac{b'^{n1}}{|B'|} \\ \frac{b'^{12}}{|B'|} & \frac{b'^{22}}{|B'|} & \cdots & \frac{b'^{n2}}{|B'|} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{b'^{1n}}{|B'|} & \frac{b'^{2n}}{|B'|} & \cdots & \frac{b'^{nn}}{|B'|} \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_n \end{pmatrix}. \tag{10}$$

From (10), we have

$$X'_i=\sum_{j=1}^{n}\frac{b'^{ji}}{|B'|}Y_j$$

$$=\frac{1}{|B'|}\sum_{j=1}^{n}\Big(b^{ji}-\sum_{\substack{k\neq j\\ l\neq i}}^{n}\varepsilon_{kl}b^{kl;ji}\Big)Y_j \tag{11}$$

$$=\frac{1}{|B'|}\Big\{\sum_{j=1}^{n}b^{ji}Y_j-\sum_{\substack{k=1\\ l\neq i}}^{n}\varepsilon_{kl}(\sum_{j\neq k}b^{kl;ji}Y_j)\Big\}$$

$$=\frac{|B|}{|B'|}\Big\{\sum_{j=1}^{n}\frac{b^{ji}}{|B|}Y_j-\sum_{\substack{k=1\\ l\neq i}}^{n}\varepsilon_{kl}\frac{b^{kl}}{|B|}\Big(\sum_{j\neq k}\frac{b^{kl;ji}}{b^{kl}}Y_j\Big)\Big\}$$

$$=\frac{|B|}{|B|\left(1-\sum_{i,j=1}^{n}\varepsilon_{ij}c_{ji}\right)}\left\{\sum_{j=1}^{n}c_{ij}Y_j-\sum_{\substack{k=1\\l\neq i}}\varepsilon_{kl}c_{lk}\left[\sum_{j\neq k}\frac{b^{kl;ji}}{b^{kl}}Y_j\right]\right\}$$

$$=\frac{1}{1-\sum_{i,j}^{n}\varepsilon_{ij}c_{ji}}\left\{X_i-\sum_{\substack{k=1\\l\neq i}}^{n}\varepsilon_{kl}c_{lk}\left(\sum_{j\neq k}\frac{b^{kl;ji}}{b^{kl}}Y_j\right)\right\}.$$

Let X_i^{kl} be $\sum_{j\neq k}\frac{b^{kl;ji}}{b^{kl}}Y_j$, where $\frac{b^{kl;ji}}{b^{kl}}$ is the ratio of the cofactor of b_{ji} in the following determinant to the determinant itself,

$$\text{(12)}\qquad \begin{vmatrix} b_{11} & \cdots & b_{1,l-1} & b_{1,l+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k-1,1} & \cdots & b_{k-1,l-1} & b_{k-1,l+1} & \cdots & b_{k-1,n} \\ b_{k+1,1} & \cdots & b_{k+1,l-1} & b_{k+1,l+1} & \cdots & b_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n,1} & \cdots & b_{n,l-1} & b_{n,l+1} & \cdots & b_{nn} \end{vmatrix}.$$

From this fact $\{X_i^{kl}\}$ is a solution of the equation

$$\text{(13)}\qquad \begin{pmatrix} b_{11} & \cdots & b_{1,l-1} & b_{1,l+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k-1,1} & \cdots & b_{k-1,l-1} & b_{k-1,l+1} & \cdots & b_{k-1,n} \\ b_{k+1,1} & \cdots & b_{k+1,l-1} & b_{k+1,l+1} & \cdots & b_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{n,l-1} & b_{n,l+1} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} X_1^{kl} \\ \cdots \\ X_{l-1}^{kl} \\ X_{l+1}^{kl} \\ \cdots \\ X_n^{kl} \end{pmatrix}=\begin{pmatrix} Y_1 \\ \cdots \\ Y_{k-1} \\ Y_{k+1} \\ \cdots \\ Y_n \end{pmatrix}.$$

Adding a trivial equation $0=0$ to this matric equation, we have

$$\text{(14)}\qquad \begin{pmatrix} b_{11} & \cdots & b_{1,l-1} & b_{1l} & b_{1,l+1} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k-1,1} & \cdots & b_{k-1,l-1} & b_{k-1,l} & b_{k-1,l+1} & & b_{k-1,n} \\ 0 & \cdots & 0 & b_{kl} & 0 & \cdots & 0 \\ b_{k+1,1} & \cdots & b_{k+1,l-1} & b_{k+1,l} & b_{k+1,l+1} & \cdots & b_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{nl} & \cdots & b_{n,l-1} & b_{nl} & b_{n,l+1} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} X_1^{kl} \\ \cdots \\ X_{l-1}^{kl} \\ 0 \\ X_{l+1}^{kl} \\ \cdots \\ X_n^{kl} \end{pmatrix}=\begin{pmatrix} Y_1 \\ \cdots \\ Y_{k-1} \\ 0 \\ Y_{k+1} \\ \cdots \\ Y_n \end{pmatrix}.$$

For the simplicity of notation we write this equation as

$$B^{kl}X^{kl}=\begin{pmatrix} Y_1 \\ \cdots \\ Y_{k-1} \\ 0 \\ Y_{k+1} \\ \cdots \\ Y_n \end{pmatrix}. \tag{15}$$

Subtracting (15) from (4) side by side, we have

$$BX-B^{kl}X^{kl}=\begin{pmatrix} Y_1 \\ Y_2 \\ \\ \cdots \\ \\ \\ Y_n \end{pmatrix}-\begin{pmatrix} Y_1 \\ \cdots \\ Y_{k-1} \\ 0 \\ Y_{k-1} \\ \cdots \\ Y_n \end{pmatrix}=\begin{pmatrix} 0 \\ \cdots \\ 0 \\ Y_k \\ 0 \\ \cdots \\ 0 \end{pmatrix},$$

hence

$$B(X-X^{kl})+(B-B^{kl})X^{kl}=\begin{pmatrix} 0 \\ \cdots \\ 0 \\ Y_k \\ 0 \\ \cdots \\ 0 \end{pmatrix},$$

hence

$$B(X-X^{kl})=\begin{pmatrix} 0 \\ \cdots \\ 0 \\ Y_k \\ 0 \\ \cdots \\ 0 \end{pmatrix}-(B-B^{kl})X^{kl} \tag{16}$$

$$
=\begin{pmatrix} 0 \\ \cdots \\ 0 \\ Y_k \\ 0 \\ \cdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \cdots & 0 \\ b_{k1} & \cdots b_{k,l-1} & 0 & b_{k,l+1} \cdots & b_{kn} \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} X_1^{kl} \\ \cdots \\ X_{l-1}^{kl} \\ 0 \\ X_{l+1}^{kl} \\ \cdots \\ X_n^{kl} \end{pmatrix}
$$

$$
=\begin{pmatrix} 0 \\ \cdots \\ 0 \\ Y_k \\ 0 \\ \cdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \sum_{m\neq l} b_{km}X_m^{kl} \\ 0 \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots\cdots \\ 0 \\ Y_k-\sum_{m\neq l} b_{km}X_m^{kl} \\ 0 \\ \cdots\cdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \alpha_k^{kl} \\ 0 \\ \cdots \\ 0 \end{pmatrix}.
$$

Thus we have

$$
(17) \qquad X-X^{kl}=\begin{pmatrix} X_1-X_1^{kl} \\ \cdots\cdots \\ X_{l-1}-X_{l-1}^{kl} \\ X_l \\ X_{l+1}-X_{l+1}^{kl} \\ \cdots\cdots \\ X_n-X_n^{kl} \end{pmatrix} = B^{-1}\begin{pmatrix} 0 \\ \cdots \\ 0 \\ \alpha_k^{kl} \\ 0 \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{11}\, c_{12} \cdots c_{1n} \\ c_{21}\, c_{22} \cdots c_{2n} \\ \\ \cdots\cdots \\ \\ c_{n1}\, c_{n2} \cdots c_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \alpha_k^{kl} \\ 0 \\ \cdots \\ 0 \end{pmatrix}
$$

$$
=\begin{pmatrix} c_{1k}\alpha_k^{kl} \\ c_{2k}\alpha_k^{kl} \\ \\ \cdots \\ \\ c_{nk}\alpha_k^{kl} \end{pmatrix},
$$

or

$$
X_1-X_1^{kl} \quad = c_{1k}\alpha_k^{kl}
$$

$$
\cdots\cdots
$$

$$
(18)\qquad
\begin{aligned}
X_{l-1}-X_{l-1}^{kl}&=c_{l-1,k}\alpha_k^{kl}\\
X_l\qquad\quad &=c_{lk}\alpha_k^{kl}\\
X_{l+1}-X_{l+1}^{kl}&=c_{l+1,k}\alpha_k^{kl}\\
\cdots\cdots\\
X_n-X_n^{kl}\quad &=c_{nk}\alpha_k^{kl}\ .
\end{aligned}
$$

Consequently

$$
(19)\qquad X_i-X_i^{kl}=\frac{c_{ik}}{c_{lk}}X_l \qquad \text{for } i=1,\cdots,l-1,\,l+1,\cdots n\,.
$$

Substituting (19) into (11) we have

$$
\begin{aligned}
X_i'-X_i&=\frac{1}{1-\sum\limits_{i,j=1}^{n}c_{ij}\varepsilon_{ji}}\Big\{-\sum_{\substack{k=1\\ l\neq i}}^{n}\varepsilon_{kl}c_{lk}(X_i^{kl}-X_i)+\sum_{k=1}^{n}\varepsilon_{ki}c_{ik}X_i\Big\}\\
&=\frac{1}{1-\sum\limits_{i,j=1}^{n}c_{ij}\varepsilon_{ji}}\Big\{\sum_{\substack{k=1\\ l\neq i}}^{n}\varepsilon_{kl}c_{lk}\frac{c_{ik}}{c_{lk}}X_l+\sum_{k=1}^{n}\varepsilon_{ki}c_{ik}X_i\Big\}\\
&=\frac{1}{1-\sum\limits_{i,j=1}^{n}c_{ij}\varepsilon_{ji}}\Big\{\sum_{k,l=1}^{n}c_{ik}\varepsilon_{kl}X_l\Big\}\,.
\end{aligned}
$$

Now the mean square error of X_i can be obtained by assuming that $E(\varepsilon_{ij})=0$, $E(\varepsilon_{ij},\ \varepsilon_{kl})=0$ for $i\neq k$ or $j\neq l$, and $E(\varepsilon_{ij}^2)=r^2a_{ij}^2$:

$$
\begin{aligned}
E(X_i'-X_i)^2&=E\left\{\frac{\sum\limits_{k,l=1}^{n}c_{ik}\,\varepsilon_{kl}X_l}{1-\sum\limits_{i,j=1}^{n}c_{ij}\,\varepsilon_{ji}}\right\}^2\\
&=E\Big[\Big(\sum_{k,l=1}^{n}c_{ik}\,\varepsilon_{kl}X_l\Big)^2\Big\{1+\sum_{i,j=1}^{n}c_{ij}\,\varepsilon_{ji}+\Big(\sum_{i,j=1}^{n}c_{ij}\,\varepsilon_{ji}\Big)^2,\cdots\Big\}^2\Big]\\
&=E\Big\{\Big(\sum_{k,l=1}^{n}c_{ik}\,\varepsilon_{kl}X_l\Big)^2\Big\}+0\Big(\frac{1}{n^3}\Big)\\
&\doteqdot\sum_{k,l=1}^{n}c_{ik}^2E(\varepsilon_{kl}^2)X_l^2\\
&=\sum_{k,l=1}^{n}c_{ik}^2r^2a_{kl}^2X_l^2\\
&=r^2\sum_{k=1}^{n}c_{ik}^2\Big(\sum_{l=1}^{n}a_{kl}^2X_l^2\Big)
\end{aligned}
$$

$$=r^2\sum_{k=1}^{n}c_{ik}^2\left(\sum_{l=1}^{n}a_{kl}X_l\right)^2\frac{\sum_{l=1}^{n}(a_{kl}X_l)^2}{\left(\sum_{l=1}^{n}a_{kl}X_l\right)^2}$$

$$=r^2\sum_{k=1}^{n}c_{ik}^2(X_k-Y_k)^2\frac{n\sigma_k^2+nm_k^2}{(nm_k)^2}$$

$$=r^2\sum_{k=1}^{n}c_{ik}^2(X_k-Y_k)^2\frac{1}{n}(C_k^2+1)$$

(where m_k, σ_k^2, C_k are the mean, the vaviance, the coefficient of variation of $a_{kl}X_l$, respectively.)

$$\leqq\frac{r^2}{n}\max(C_k^2+1)\sum_{k=1}^{n}c_{ik}^2(X_k-Y_k)^2$$

$$\leqq\frac{r^2}{n}(C^2+1)\left\{\sum_{k=1}^{n}c_{ik}(X_k-Y_k)\right\}^2$$

$$=\frac{r^2}{n}(C^2+1)\left\{\left(\sum_{k=1}^{n}c_{ik}Y_k\right)\frac{\sum_{k=1}^{n}c_{ik}Y_k\frac{X_k}{Y_k}}{\sum_{k=1}^{n}c_{ik}Y_k}-\sum_{k=1}^{n}c_{ik}Y_k\right\}^2$$

$$=\frac{r^2}{n}(C^2+1)X_i^2(M_i-1)^2,$$

where C is the maximum of C_k, and M_i is a weighted mean of $\frac{X_k}{Y_k}$ and is not so different from $\frac{\sum_{k=1}^{n}X_k}{\sum_{k=1}^{n}Y_k}=M$.

From the above the relative error of X_i can be estimated by $\frac{r}{\sqrt{n}}\sqrt{C^2+1}\,(M-1)$ where C is a number near 1, and M is not so large, thus the dominant value in this formula is n, when n is large. This means that the larger the number of industries, the smaller the error of output caused by the technical coefficients errors.

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