ON INTERTEMPORAL EFFICIENCY CONDITIONS OF CAPITAL ACCUMULATION (I)

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1. Introduction

Samuelson and Solow [6] has handled the problem of intertemporal production efficiency conditions for a general dynamic capital model, which is closely connected with such a model as the Ramsey one-good model, the von Neumann general equilibrium model or the Leontief dynamic system. This problem seems to have received little attention by the classical economists, and [6] may be the first handling it. They there seeks the conditions, efficient paths of dynamic development must fulfill, and shows that this can be represented by the so-called "envelope rule". The main results for a closed economy, explicitly or implicitly formulated in [6], may be summarized as follows[1]:

Let there be n distinct commodities in the economy, and $x(t)=(x_1(t), \cdots, x_n(t))$ be stocks of each commodity at time t. We get the stock vector

$$x(t+1)=(x_1(t+1), \cdots, x_n(t+1)),$$

by allocating the initial stock vector $x(t)$ among the possible technological processes of this economy for the time interval $[t, t+1]$ in the optimal ways. We shall represent this stock input-output relation by the transformation function:

$$T[x_1(t), \cdots, x_n(t); x_1(t+1), \cdots, x_n(t+1)]=0,$$

or in the vector notation,

$$T[x(t), x(t+1)]=0,$$

which is subject to constant returns to scale and defines a convex frontier. This stock vector will be used as inputs for the production in the next time interval $[t+1, t+2]$ and produce a stock vector $x(t+2)$ at time $t+2$. In repeating this procedure h-times, we can get a path

1) Acknowledgement is due to Professors Samuelson and Solow, who have kindly permitted the author to refer to their unpublished paper [6].

of stock vectors

$$(x(t), \cdots, x(t+h)) .$$

This path is said to be intertemporally *efficient*, if the amount of the stock $x_i(t+h)$ of any commodity at time $t+h$ cannot be better off without sacrificing at least one of the stocks of other commodities.

Then we obtain the following properties.

A: If a path $(x(t), x(t+1))$ is efficient, i.e.

$$T(x(t), x(t+1))=0$$

then there exists a stock vector $x(t+2)$ such that

$$(x(t), x(t+1), x(t+2))$$

is an efficient path.

B: A path $(x(t), x(t+1), x(t+2))$ is efficient, if and only if the marginal rate of substitution between any two commodities in $x(t+1)$ regarded as outputs of the production with the initial stock vector $x(t)$ must exactly equal their marginal rate of substitution regarded as inputs for the production of the stock vector $x(t+2)$.

C: A path $(x(t), x(t+1), \cdots, x(t+h))$ is efficient if, and only if

$$(x(t+\tau), x(t+\tau+1), x(t+\tau+2))$$

is an efficient path for any $\tau=0, 1, \cdots, h-2$.

These results, combined with the notion of balanced growth as in Samuelson and Solow [7], will imply the following

D: There exists at least an efficient path

$$(x(t), x(t+1), \cdots, x(t+h))$$

such that

$$x(t+\tau+1)=\lambda x(t+\tau) \qquad (\tau=0, 1, \cdots, h-1)$$

where λ is a positive constant.

We can easily guess that the statement B, the intertemporal generalization of the rule of equality of marginal rates of substitution, holds good. It seems, however, that the other statements are rather doubtful. Especially, from the viewpoint of an intertemporal optimality in general, we usually think that a path, even if it may be optimal at each time in considering before and after, would not necessarily be optimal for a long time.

In what follows, therefore, we shall be concerned with a settheoretical formulation of the above problem, and the assumptions underlying the analysis will be explicitly explained.

2. Transformation sets

We suppose that there are n commodities $\{1, \cdots, n\}$ in the economy. Labor will be included as a commodity. By a *stock vector* $x=(x_1, \cdots, x_n)$ is meant a non-negative n-dimensional vector, the i-th component of which is the available stock of the i-th commodity for the economy. The set of all stock vectors, i.e. the non-negative octant of the n-dimensional euclidean space, will be denoted by Ω.

Let there be a stock vector $x(t)$ at time t and it be allocated among the technological processes of the economy. Then the possibility for the stock vector $x(t+1)$ at time $t+1$ is limited by the limitations on production by this economy in the time interval $[t, t+1]$. We shall express these limitations by the *transformation set* T_t in the following way:

A transformation set T_t at $[t, t+1]$ is a subset of the product space $\Omega \times \Omega$, i.e. a set of non-negative $2n$-vectors, and when there is a stock $x(t)$ at time t, the set of all possible output stock vectors $x(t+1)$ at time $t+1$, after the production using the initial resources $x(t)$, is precisely $T_t(x(t))$,

where $$T_t(x(t))=\{x(t+1); (x(t), x(t+1)) \in T_t\} .$$

In other words, the output stock vector $x(t+1)$ can be produced from the initial stock vector $x(t)$ if, and only if $x(t)$ belongs to the set $T_t^*(x(t+1))$,

where $$T_t^*(x(t+1))=\{x(t); (x(t), x(t+1)) \in T_t\} .$$

We suppose that, for any time t, a transformation set T_t fulfills the following conditions[2]:

T1. For any stock vector $x^0 \in \Omega$, $T_t(x^0)$ is a non-empty and compact subset of Ω.

T2. If $x^0 \leq y^0$ for two stock vectors x^0, y^0 and $x^1 \in T_t(x^0)$, then there

2) As usual, we shall denote, for two vectors x and y,

$$x \geqq y, \quad \text{when} \quad x_i \geqq y_i \qquad (i=1, \cdots, n),$$
$$x \geq y, \quad \text{when} \quad x_i \geq y_i \qquad (i=1, \cdots, n)$$
and $x_i > y_i$ at least for an index i,
$$x > y, \quad \text{when} \quad x_i > y_i \qquad (i=1, \cdots, n)$$

exists a stock vector y^1 such that

$$x^1 \leq y^1 \quad \text{and} \quad y^1 \in T_t(y^0) .$$

T3. If $x^1 \leq y^1$ for two stock vectors x^1, y^1 and $x^0 \in T_t(x^1)$, then there exists a stock vector y^0 such that

$$x^0 \leq y^0 \quad \text{and} \quad y^0 \in T_t^*(y^1) .$$

T4. T_t is a convex set, i.e. if (x^0, x^1) and (y^0, y^1) belong to T_t, then

$$(\lambda x^0 + (1-\lambda)y^0,\ \lambda x^1 + (1-\lambda)y^1)$$

belongs to T_t for any $0 < \lambda < 1$.

As special cases of T4, we can get the usual laws of diminishing returns (the convexity of the set $T_t(x^0)$ for any stock vector x^0) and diminishing rates of substitution (the convexity of the set $T_t{}^*(x^1)$ for any stock vector x^1). The converse of this statement, however, cannot be assured.

3. Possible paths and inter-temporal efficiency

A sequence of stock vectors $(x^0, x^1, \cdots, x^h)$ is said to be a *possible path*, or simply a *path*, if we have

$$(x^t, x^{t+1}) \in T_t$$

or

$$x^{t+1} \in T_t(x^t) \qquad (t=0, 1, \cdots, h-1).$$

A stock vector x^h is said to be *possible* from a stock vector x^0 after the h-th production, if there exists a possible path

$$(x^0, \bar{x}^1, \cdots, \bar{x}^h)$$

such that $\bar{x}^h = x^h$, and the set of all possible stock vectors from the stock vector x^0 after the h-th production is denoted by $T^h(x^0)$:

$$T^h(x^0) = \{x^h; (x^0, x^1, \cdots, x^h) \text{ is possible}\}$$

For $h=1$, we have

$$T^1(x^0) = T(x^0) .$$

The set $T^h(x^0)$ is convex, compact and non-empty for any stock vector x^0.

For a set X of n-vectors, an element $\bar{x}$ is said to be *efficient* in X, if

(a) $\bar{x} \in X$

(b) $\bar{x} \leq x$ for no $x \in X$.

The set of efficient elements of X is denoted by $E(X)$.

A possible path $(x^0, x^1, \cdots x^h)$ is said to be *efficient*, if x^h is in $E^h(x^0)$, where $E^h(x^0)$ is the set of all efficient elements in $T^h(x^0)$:

$$E^h(x^0) = E(T^h(x^0)) .$$

4. Efficiency conditions for the case $h=1$

For a stock vector x^0, the set of all efficient vectors in $T(x^0)$ is denoted by $E(x^0)$, and a possible path (x^0, x^1) is efficient if and only if $x^1 \in E(x^0)$.

THEOREM 1. *$E(x^0)$ is non-empty for any stock vector x^0.*

PROOF. For a positive vector $p=(p_1, \cdots, p_n)$, and a positive number c, let

$$K_c = \{x ; x \in \Omega, p \cdot x \geqq c\} ,$$

where $p \cdot x$ denotes the inner product of p and x:

$$p \cdot x = \sum_{i=1}^{n} p_i \cdot x_i .$$

If $c > c'$, then $K_c \subset K_{c'}$.

Since $X = T(x^0)$ is compact in Ω,

$$K_{\bar{c}} \cap X = \phi$$

for a sufficiently large number $\bar{c}$, where ϕ is the non-empty set. Then

$$c_0 = \inf \{c ; K_c \cap X = \phi\}$$

is positive and finite.

If $K_{c_0} \cap X = \phi$, then

$$\text{dis}\,(K_{c_0}, X) = \inf \{|x-y| ; x \in X, y \in K_{c_0}\} > 0 ,$$

where $|v|$ is the length of a vector v:

$$|v| = \sqrt{\sum_{i=1}^{n} v_i^2} .$$

Hence there exists a positive number ε such that

$$K_{c_0-\varepsilon} \cap X = \phi ,$$

which is a contradiction. Therefore, we must have

$$K_{c_0} \cap X \neq \phi .$$

Let $\bar{x}$ be an element of $K_{c_0} \cap X$, then

$$p \cdot \bar{x} \geqq c_0 .$$

If $p \cdot x = c > c_0$ for some $x \in X$, then

$$K_c \cap X \neq \phi$$

and $c > c_0$, which contradicts to the definition of c_0. Therefore

$$p \cdot x \leqq c_0 \qquad \text{for all} \quad x \in X.$$

If $\bar{x}$ is not efficient in X, then there would exist a vector $x \in X$ such that

$$x \geq \bar{x} .$$

Since $p > 0$, we have

$$p \cdot x > p \cdot \bar{x}$$

which is a contradiction, q.e.d.

By a price vector p is meant a non-negative n-vector. For a price vector p we shall define sets of indices $Z(p)$ and $P(p)$ as follows:

$$Z(p) = \{i \,;\, p_i = 0\} \quad \text{and} \quad P(p) = \{i \,;\, p_i > 0\} .$$

A finite sequence of price vectors $(p^1, \cdots, p^s)$ is said to be a *normal sequence of price vectors*, if the following conditions are satisfied:

$$Z(p^k) \subset P(p^{k+1}) \cup \cdots \cup P(p^s) \qquad (k = 1, 2, \cdots, s-1).$$

For a normal sequence of price vectors $(p^1, \cdots, p^s)$, we shall define subsets of $T(x^0)$ as follows:

$$E(x^0 ; p^1) = \left\{ \bar{x} \,;\, \bar{x} \in T(x^0),\ p^1 \cdot \bar{x} = \max_{x \in T(x^0)} p^1 \cdot x \right\}$$

$$E(x^0 ; p^1, p^2) = \left\{ \bar{x} \,;\, \bar{x} \in E(x^0 ; p^1),\ p^2 \cdot \bar{x} = \max_{x \in E(x^0; p^1)} p^2 \cdot x \right\}$$

. . . .

$$E(x^0 ; p^1, \cdots, p^s)$$

$$= \left\{ \bar{x} \,;\, \bar{x} \in E(\bar{x}^0 ; p^1, \cdots, p^{s-1}),\ p^s \cdot \bar{x} = \max_{x \in E(x^0;\, p^1, \cdots, p^{s-1})} p^s \cdot x \right\} .$$

Similarly, subsets $E^*(x^1; p^1), \cdots, E^*(x^1; p^1, \cdots, p^s)$ of $T^*(x^1)$ are defined as follows:

$$E^*(x^1; p^1) = \left\{\bar{x}; \bar{x} \in T^*(x^1),\ p^1 \cdot \bar{x} = \min_{x \in T^*(x^1)} p^1 \cdot x\right\}$$

$$E^*(x^1; p^1, p^2) = \left\{\bar{x}; \bar{x} \in E^*(x^1; p^1),\ p^2 \cdot \bar{x} = \min_{x \in E^*(x^1; p^1)} p^2 \cdot x\right\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$E^*(x^1; p^1, \cdots, p^s)$$

$$= \left\{\bar{x}; \bar{x} \in E^*(x^1; p^1, \cdots, p^{s-1}) p^s \cdot \bar{x} = \min_{x \in E^*(x^1; p^1, \cdots, p^{s-1})} p^s \cdot x\right\}.$$

THEOREM 2. *A stock vector $\bar{x}$ is efficient in $T(x^0)$ if and only if there exists a normal sequence of price vectors $(p^1, \cdots, p^s)$ such that*

$$\bar{x} \in E(x^0; p^1, \cdots, p^s).$$

In other words,

$$E(X^0) = \bigcup_{(p^1, \cdots, p^s) \text{ is a normal sequence}} E(x^0; p^1, \cdots, p^s).$$

PROOF. (1) Let $\bar{x} \in E(x^0; p^1, \cdots, p^s)$ for a normal sequence $(p^1, \cdots, p^s)$. If $\bar{x} \notin E(x^0)$, we shall show this leads to a contradiction. In that case, there would exists an element $x \in T(x^0)$ such that

$$x \geq \bar{x}.$$

Let i be an index such that $x_i > \bar{x}_i$ and k be such that

$$p^1 \cdot x = p^1 \cdot \bar{x}, \cdots, p^{k-1} \cdot x = p^{k-1} \cdot \bar{x} \quad \text{and} \quad P_i^k > 0.$$

Then
$$p^k \cdot x > p^k \cdot x^0.$$

Since $x \in E(x^0; p^1, \cdots, p^{k-1})$, this is a contradiction.

(2) When $\bar{x} \in E(x^0)$, then we shall show that there exists a price vector p^1 such that

$$\bar{x} \in E(x^0; p^1).$$

Let
$$K = \{x; \bar{x} \leqq x \leqq \xi\}$$

where ξ is a positive vector such that

$$x < \xi$$

for every $x \in T(x^0)$. The existence of ξ is implied from the compactness of $T(x^0)$.

Since K and $T(x^0)$ are convex in Ω, and

$$K \cap T(x^0) = \{\bar{x}\} \ ,$$

there exists a non-zero vector p^1 so that

$$p^1 \cdot x \leqq p^1 \cdot \bar{x} \qquad \text{for} \quad x \in T(x^0).$$

$$p^1 \cdot x \geqq p^1 \cdot \bar{x} \qquad \text{for} \quad x \in K.$$

In considering a vector x in K, for which

$$\bar{x}_i < x_i < \xi_i$$

$$\bar{x}_k = x_k \qquad (k \neq i),$$

we concluce that $p_i^1 \geqq 0$ for $i = 1, 2, \cdots, n$.

Hence $\bar{x} \in E(x^0; p^1)$.

Since $E(x^0; p^1)$ is contained in a $(n-1)$-dimensional space $\{x;\ p^1 \cdot x = p^1 \cdot \bar{x}\}$, it is easily seen that there exists a price vector P^2 such that

$$N(p^2) \not\supset N(p^1)$$

and

$$\bar{x} \in E(x^0; p^1, p^2) \ .$$

In repeating this procedure we can obtain the desired normal sequence of price vectors, q.e.d.

COROLLARY. *If* $\bar{x} \in T(x^0)$ *and*

$$p \cdot \bar{x} = \max_{x \in T(x^0)} p \cdot x$$

for some positive vector $p > 0$, *then* $\bar{x}$ *is efficient in* $T(x^0)$.

When we define, for a stock vector x^1, the set $E^*(x^1)$ as follows:

$$E^* = \{\bar{x};\ \bar{x} \in T^*(x^1),\ \bar{x} \geq x \text{ for no } x \in T^*(x^1)\} \ ,$$

then we obtain the following theorem, the proof of which can be done in the similar way as in the proof of theorem 2.

THEOREM 3. $E^*(x^1) = \bigcup_{(p^1, \cdots, p^s) \text{ is a normal sequence}} E^*(x^1; p^1, \cdots, p^s).$

COROLLARY. *If* $\bar{x} \in T^*(x^1)$ *and*

$$p \cdot \bar{x} = \min_{x \in T^*(x^1)} p \cdot x$$

with a positive vector $p > 0$, *then* $\bar{x} \in E^*(x^1)$.

5. Efficiency conditions for the case $h=2$

For a stock vector x^0, $T^2(x^0)$ is the set of all stock vectors x^2 where there exists a stock vector x^1 so that (x^0, x^1, x^2) is a possible path, and $E^2(x^0)$ is the set of all efficient vectors in $T^2(x^0)$. A path (x^0, x^1, x^2) is efficient if $x^2 \in E(x^0)$. If a path (x^0, x^1, x^2) is efficient, then (x^0, x^1) and (x^1, x^2) are efficient.

THEOREM 5. *A path (x^0, x^1, x^2) is efficient, if and only if there exists a normal sequence of price vectors $(p^1, \cdots, p^s)$ such that*

$$x^1 \in E(x^0; p^1, \cdots, p^s) \cap E^*(x^2; p^1, \cdots, p^s) .$$

PROOF. Let (x^0, x^1, x^2) be efficient, K the smallest convex set containing $E(x^0)$ and $\{x; 0 \leqq x \leqq x^1\}$, and L the smallest convex set containing $E^*(x^2)$ and $\{x; x^1 \leqq x\}$. Then K and L have no inner point in common. Therefore, there exists a price vector p^1 such that

$$x^1 \in E(x^0; p^1) \cap E^*(x^2; p^1) .$$

In repeating the same process as in the proof of theorem 2, we can conclude the existence of the desired normal sequence of price vectors.

Let, for a path (x^0, x^1, x^2), there exist a normal sequence of price vectors $(p^1, \cdots, p^s)$ such that

$$x^1 \in E(x^0; p^1, \cdots, p^s) \cap E^*(x^2; p^1, \cdots, p^s) .$$

Then $E(x^0)$ and $E^*(x^2)$ have no inner point in common. If we suppose that (x^0, x^1, x^2) is not efficient, then we shall show this would lead to a contradiction. In that case, there would be stock vectors $\bar{x}^1$ and $\bar{x}^2$ so that $(x^0, \bar{x}^1, \bar{x}^2)$ is a possible path and $\bar{x}^2 \geq x^2$. Then there exists a stock vector y^1 such that (x^0, y^1, x^2) also is possible and $y^1 \leq \bar{x}^1$. This would contradict to our assumption, q.e.d.

COROLLARY. *If, for a path (x^0, x^1, x^2), there exists a positive vector $p > 0$ such that*

$$p \cdot x^1 = \max_{x \in T(x^0)} p \cdot x = \min_{x \in T^*(x^2)} p \cdot x ,$$

then (x^0, x^1, x^2) is an efficient path.

6. Efficiency conditions for an arbitrary h

A sequence of vectors $(x^0, x^1, \cdots, x^h)$ is possible, if

$$(x^t, x^{t+1}) \in T_t$$

for any $t=0, 1, \cdots, h-1$. $T^h(x^0)$ is the set of all vectors x^h such that there exists a set of vectors $x^1, \cdots, x^{h-1}$ where $(x^0, x^1, \cdots, x^{h-1}, x^h)$ is possible. $E^h(x^0)$ is the set of all efficient vectors in $T^h(x^0)$, and a path $(x^0, x^1, \cdots, x^h)$ is said to be efficient, if $x^h \in E^h(x^0)$.

THEOREM 6. *$E^h(x^0)$ is non-empty for any stock vector x^0. That is, there exists at least an efficient path $(x^0, x^1, \cdots, x^h)$ for any integer h.*

Proof of this theorem can be done in the similar way as in the proof of theorem 1.

THEOREM 7. *If a path $(x^0, x^1, \cdots, x^h)$ is efficient, then each subpath*

$$(x^r, x^{r+1}, \cdots, x^s)$$

also is an efficient path for any $0 \leqq r < s \leqq h$.

PROOF. If there exists a subpath

$$(x^r, x^{r+1}, \cdots, x^s)$$

which is not efficient, then there would exist a stock vector $\bar{x}^s$ in $T^{s-r}(x^r)$ so that

$$\bar{x}^s \geq x^s .$$

Then there exists $\bar{x}^h$ such that $\bar{x}^h \geq x^h$, $\bar{x}^h \in T^{h-s}(\bar{x}^s)$. Therefore

$$\bar{x}^h \in T^h(x^0) \quad \text{and} \quad \bar{x}^h \geq x^h ,$$

which contradicts to the assumption $x^h \in E^h(x^0)$, q.e.d.

COROLLARY. *If a path $(x^0, x^1, \cdots, x^h)$ is efficient, then*

$$(x^t, x^{t+1}, x^{t+2})$$

is efficient for any $t=0, 1, \cdots, h-2$.

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