ON OPTIMUM CHARACTER OF VON NEUMANN'S MONTE CARLO MODEL*

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0. Introduction

In the former paper [1] the author discussed an application of the Monte Carlo method to the solution of linear simultaneous equations. The process developed there was a variation of J. von Neumann's splitting technique. The mean and variance of our estimate and those of the total number of random digits required for getting an estimate are given there. In this paper we shall investigate the structure of splitting technique developed by von Neumann in its original form. As a result of this it will be seen that the splitting technique has an optimum character as a practical procedure of estimation.

1. J. von Neumann's splitting technique.

In the paper by Kahn and Harris [2], splitting technique is described as follows;

"The concentration of neutrons at the point in phase space denoted by (x, α, γ) is given by a function $\psi(x, \alpha, \gamma)$ where x is the position coordinate, α the energy, and γ the cosine of the angle with the normal to the slab. We define a function $\phi(x, \alpha, \gamma)$ such that $\phi(x, \alpha, \gamma)$ is the probability that a particle (x, α, γ) will eventually be transmitted. The function ϕ represents the importance of the region at (x, α, γ). Finally, there is a transition function $f(x, \alpha, \gamma; x', \alpha', \gamma')$ which measure the rate at which particles flow from (x, α, γ) to (x', α', γ'). The splitting technique, performed in optimum fashion, would be somewhat as follows. A set of surfaces would be defined by the equations:

$$\phi(x, \alpha, \gamma)=2^{-\sigma},$$

where σ has values running from one to some integer n. In other

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words, surfaces of constant importance are used rather than surfaces of fixed values of x. Whenever a particle passes from a less important to a more important region, it is split into two. Each of the resulting particles is given one-half the weight of the original particle and is treated independently from then on. When the passage is from the more important to a less important region, a game of chance is played in which the particle has a 50-percent chance of being eliminated and a 50-percent chance of being allowed to survive with double its original weight. It is easy to see that all the particles that get through will have the weight 2^{-n} and if N histories are started originally and k particles get through then an estimate of the probability of transmission is $(k/N)2^{-n}$".

Now, for the sake of simplicity, we shall consider here a set of discrete states running 1 to n and a special state k representing one of traps (state of getting through the slab).

Let p_{ij} be the probability that a particle jumps from the i-th to the j-th state.

Let p_i be the probability that a particle starts in the i-th state (probability of a particle beeing generated at the i-th state).

Let ϕ_i be the probability of eventually getting to the k-th state on starting from the i-th state. Then it is required to get an efficient estimate of

$$T=\sum_{i=1}^{n} p_i\phi_i+p_k\,.$$

T represents the probability of transmission of a particle, or a probability of a particle being generated and eventually trapped in the k-th state.

To see the optimum character of splitting technique we shall study first so called variable weight method.

2. Variable weight method.

As our purpose is to estimate T, we may make use of another artificial Markoff process corresponding to the original one with some desired properties.

Let p^*_{ij}, p^*_i be the quantities of our artificial process corresponding to the original p_{ij}, p_i. When a particle is generated at the i-th state (with probability p_i^*) it is given the weight p_i/p_i^*, and when a particle is transmitted from the i-th to the j-th state its weight is multiplied by the factor p_{ij}/p_{ij}^*. We transmit a particle by this artificial process and get an estimate $\hat{T}_w$ of T which takes as its value the weight of particle when the particle is eventually trapped at the k-th state and zero otherwise. To get the mean and variance of $\hat{T}_w$, we consider random variables $X(ik)$ which represent the weight of a particle which starts from the i-th state with weight 1 and is trapped eventually at the k-th state. Assuming the existence of $EX(ik)$, $EX^2(ik)(i=1, 2, \cdots, n)$, we get the following formulae ;

$$EX(ik)=\sum_{j=1}^{n} p_{ij}^*\left[\frac{p_{ij}}{p_{ij}^*}\right]EX(jk)+p_{ik}^*\left[\frac{p_{ik}}{p_{ik}^*}\right]$$

$$EX^2(ik)=\sum_{j=1}^{n} p_{ij}^*\left[\frac{p_{ij}}{p_{ij}^*}\right]^2 EX^2(jk)+p_{ik}^*\left[\frac{p_{ik}}{p_{ik}^*}\right]^2; \qquad i=1, 2, \cdots, n$$

or in the matrix notation

$$(EX(ik))=(I-P)^{-1}(p_{ik})\equiv(\phi_{ij})(p_{ik})=(\phi_i)$$

$$(EX^2(ik))=\left(I-\left(\left[\frac{p_{ij}}{p_{ij}^*}\right]p_{ij}\right)\right)^{-1}\left(\left[\frac{p_{ik}}{p_{ik}^*}\right]p_{ik}\right).$$

When we denote the variance of $X(ik)$ by $D^2X(ik)$ we get

$$D^2X(ik)=\sum_{j=1}^{n} p_{ij}^*\left[\frac{p_{ij}}{p_{ij}^*}\right]^2 D^2X(jk)+\sum_{j=1}^{n} p_{ij}^*\left[\frac{p_{ij}}{p_{ij}^*}\right]^2\phi_j^2+p_{ik}^*\left[\frac{p_{ik}}{p_{ik}^*}\right]^2-\phi_i^2$$

or

$$D^2X(ik)=\sum_{j=1}^{n} p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]D^2X(jk)+\sum_{j=1}^{k} p_{ij}^*\left(\left[\frac{p_{ij}}{p_{ij}^*}\right]\phi_j-\phi_i\right)^2$$

where $\sum_{j=1}^{k}$ denotes the summation running over $j=1, 2, \cdots, n$ and k. The formula can be written in the matrix notation as

$$(D^2X(ik))=\left(I-\left(p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]\right)\right)^{-1}\left(\sum_{j=1}^{k} p_{ij}^*\left(\left[\frac{p_{ij}}{p_{ij}^*}\right]\phi_j-\phi_i\right)^2\right).$$

As for $\hat{T}_w$ we have

$$E(\hat{T}_w)=\sum_{i=1}^{n} p_i^*\left[\frac{p_i}{p_i^*}\right]EX(ik)+p_k^*\left[\frac{p_k}{p_k^*}\right]=\sum_{i=1}^{n} p_i\phi_i+p_k=T$$

$$E(\hat{T}_w^2)=\sum_{i=1}^{n} p_i^*\left[\frac{p_i}{p_i^*}\right]^2 EX^2(ik)+p_k^*\left[\frac{p_k}{p_k^*}\right]^2$$

$$=\sum_{i=1}^{n} p_i\left[\frac{p_i}{p_i^*}\right] EX^2(ik)+p_k\left[\frac{p_k}{p_k^*}\right]$$

$$D^2(\hat{T}_w)=\sum_{i=1}^{n} p_i^*\left[\frac{p_i}{p_i^*}\right]^2 D^2X(ik)+\sum_{i=1}^{k} p_i^*\left[\frac{p_i}{p_i^*}\right]^2\phi_i^2-T^2$$

$$=\sum_{i=1}^{n} p_i\left[\frac{p_i}{p_i^*}\right] D^2X(ik)+\sum_{i=1}^{k} p_i^*\left(\left[\frac{p_i}{p_i^*}\right]\phi_i-T\right)^2$$

$$=\left(p_i\left[\frac{p_i}{p_i^*}\right]\right)'\left(I-\left(p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]\right)\right)^{-1}\left(\sum_{j=1}^{k} p_{ij}^*\left(\left[\frac{p_{ij}}{p_{ij}^*}\right]\phi_j-\phi_i\right)^2\right)$$

$$+\sum_{i=1}^{k} p_i^*\left(\left[\frac{p_i}{p_i^*}\right]\phi_i-T\right)^2 .$$

It follows from the last equation that $D^2(\hat{T}_w)$ is equal to zero if and only if

$$\frac{p_{ij}}{p_{ij}^*}=\frac{\phi_i}{\phi_j}, \quad \frac{p_i}{p_i^*}=\frac{T}{\phi_i} .$$

In practical applications only the values p_{ij}s and p_is are given and we do not know the exact values of ϕ_is, so we take as p_{ij}^*s and p_i^*s estimated values of $p_{ij}\frac{\phi_j}{\phi_i}$s and $p_{ij}\frac{\phi_i}{T}$s obtained from the former experience, respectively. In this case the weight factors $\frac{p_{ij}}{p_{ij}^*}$ and $\frac{p_i}{p_i^*}$ must be used throughout the history of a particle and this is a too labourious task to make this process practicable. However there is a rescue for this default of the variable weight method in splitting technique.

3. Splitting technique in the purest form.

Corresponding to the ideal variable weight process, the following splitting process is considered. We trace a history of a particle following the original p_is and p_{ij}s, but when the particle is generated at the i-th state we split it into r_i independent particles each with weight $\frac{T}{\phi_i}$ and when a particle is transmitted from the i-th to the j-th state we split it into r_{ij} independent particles each with weight multiplied by the factor $\frac{\phi_i}{\phi_j}$, where r_j and r_{ij} are random variables independent of any other variables and with means $Er_i=\frac{\phi_i}{T}$, $Er_{ij}=\frac{\phi_j}{\phi_i}$ and variances σ_i^2, σ_{ij}^2 respectively. We take as our estimate of T the total weight $\hat{T}_s$ of descendants, of a particle, trapped at the k-th state. It is obvious that in this case every decsendant trapped at the k-th state has a weight equal to T. We shall observe the mean and variance of $\hat{T}_s$. We denote by $X(ik)$ the random variable which represents the total number of descendants eventually trapped at the k-th state starting from a particle at the i-th state. Assuming the existence of $EX(ik)$ and $EX^2(ik)$ we obtain the following formulae:

$$EX(ik)=\sum_{j=1}^{n} p_{ij}\frac{\phi_j}{\phi_i}EX(jk)+p_{ik}\frac{1}{\phi_i}$$

$$EX^2(ik)=\sum_{j=1}^{k} p_{ij}E\left(\sum_{\nu=1}^{r_{ij}} X(jk)(\omega_\nu)\right)^2$$

$$=\sum_{j=1}^{k} p_{ij}\underset{r_{ij}}{E}(r_{ij}EX^2(jk)+r_{ij}(r_{ij}-1)E^2X(jk))$$

$$=\sum_{j=1}^{k} p_{ij}\frac{\phi_j}{\phi_i}EX^2(jk)-\sum_{j=1}^{k} p_{ij}\frac{\phi_j}{\phi_i}E^2X(jk)$$

$$+\sum_{j=1}^{k} p_{ij}E(r_{ij}^2)E^2X(jk)\,.$$

From the first formula we have

$$(EX(ik))=\left(I-\left(p_{ij}\frac{\phi_j}{\phi_i}\right)\right)^{-1}\left(\frac{p_{ik}}{\phi_i}\right)$$

$$=\left(\frac{\phi_j}{\phi_i}\phi_{ij}\right)\left(\frac{p_{ik}}{\phi_i}\right)=\left(\frac{1}{\phi_i}\sum_{j=1}^{n}\phi_{ij}\phi_j\frac{p_{jk}}{\phi_j}\right)$$

$$=\left(\frac{1}{\phi_i}\sum_{j=1}^{n}\phi_{ij}p_{ik}\right)=(1)\,.$$

For the variance $D^2X(ik)$ of $X(ik)$ we have

$$D^2X(ik)=\sum_{j=1}^{n} p_{ij}\frac{\phi_j}{\phi_i}D^2X(jk)+\sum_{j=1}^{k} p_{ij}E(r_{ij}^2)-1$$

$$=\sum_{j=1}^{n} p_{ij}\frac{\phi_j}{\phi_i}D^2X(jk)+\sum_{j=1}^{k} p_{ij}\sigma_{ij}^2+\sum_{j=1}^{k} p_{ij}\left(\frac{\phi_j}{\phi_i}-1\right)^2$$

$$(D^2X(ik) = \left(I - \left(p_{ij}\frac{\phi_j}{\phi_i}\right)\right)^{-1}\left(\sum_{j=1}^{k} p_{ij}E(r_{ij}-1)^2\right).$$

As for $\hat{T}_s$ we have

$$E\hat{T}_s = \sum_{i=1}^{k} p_i \frac{T}{\phi_i} E\left(\sum_{\nu=1}^{r_i} X(ik)(\omega_\nu)\right)$$

$$= \sum_{i=1}^{k} p_i E_{r_i}(r_i EX(ik)) = \sum_{i=1}^{k} p_i \frac{\phi_i}{T} = 1$$

$$E\hat{T}_s^2 = \sum_{i=1}^{k} p_i E\left(\sum_{\nu=1}^{r_i} X(ik)(\omega_\nu)\right)^2$$

$$= \sum_{i=1}^{k} p_i E_{r_i}(r_i EX^2(ik) + r_i^2 E^2X(ik) - r_i E^2X(ik))$$

$$= \sum_{i=1}^{k} p_i \frac{\phi_i}{T} EX^2(ik) - \sum_{i=1}^{k} p_i \frac{\phi_i}{T} E^2X(ik) + \sum_{i=1}^{k} p_i E(r_i^2) E^2X(ik)$$

$$D^2(\hat{T}_s) = \sum_{i=1}^{n} \frac{p_i\phi_i}{T} D^2X(ik) + \sum_{i=1}^{k} p_i E(r_i^2) - 1$$

$$= \sum_{i=1}^{n} \frac{p_i\phi_i}{T} D^2X(ik) + \sum_{i=1}^{k} p_i\sigma_i^2 + \sum_{i=1}^{k} p_i\left(\frac{\phi_i}{T} - 1\right)^2$$

$$= \left(\frac{p_i\phi_i}{T}\right)'\left(I - \left(p_{ij}\frac{\phi_j}{\phi_i}\right)\right)^{-1}\left(\sum_{j=1}^{k} p_{ij}\sigma_{ij}^2 + \sum_{j=1}^{k} p_{ij}\left(\frac{\phi_j}{\phi_i} - 1\right)^2\right)$$

$$+ \sum_{i=1}^{k} p_i\sigma_i^2 + \sum_{i=1}^{k} p_i\left(\frac{\phi_i}{T} - 1\right)^2.$$

Thus it can be seen from the last formula, that even in this ideal case the splitting technique introduces a positive variance except for the trivial case of $\phi_i = T$ and $\sigma_{ij}^2 = \sigma_i^2 = 0$ for all is and js.

4. Splitting technique in the general form.

In this section we shall treat the splitting technique in its general form.

Let us denote by p_i^* and p_{ij}^* the estimated values of $p_i\frac{\phi_i}{T}$ and $p_{ij}\frac{\phi_j}{\phi_i}$ respectively.

We trace a history of a particle following the original p_is and p_{ij}s, but when a particle of weight unity is generated at the i-th state it is split into r_i independent particles with weight p_i/p_i^*, and when a particle is transmitted from the i-th state to the j-th state it is split into r_{ij} independent particles each with its weight multiplied by the factor $\frac{p_{ij}}{p^*_{ij}}$, where r_i and r_{ij} are random variables which are independent of any other variables and with means $Er_i=\frac{p^*_i}{p_i}$, $Er_{ij}=\frac{p^*_{ij}}{p_{ij}}$ and variances σ^2_i, σ^2_{ij} respectively. Thus we get a splitting process corresponding to the variable weight process with p^*_is and p^*_{ij}s. In this case our estimate $\hat{T}_s$ of T is given by the total weight of descendants eventually trapped at the k-th state from a particle generated at some state.

Let us denote by $X(ik)$ the total weight of descendents trapped eventually at the k-th state from a particle starting from the i-th state with weight unity. Then we get the following formulae;

$$EX(ik)=\sum_{j=1}^{k}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]E\Big(\sum_{\nu=1}^{r_{ij}}X(jk)(\omega_\nu)\Big)$$

$$=\sum_{j=1}^{n}p_{ij}EX(jk)+p_{ik}$$

or

$$(EX(ik))=(I-(p_{ij}))^{-1}(p_{ik})=(\phi_i)\,,$$

$$EX^2(ik)=\sum_{j=1}^{k}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]^2E\Big(\sum_{\nu=1}^{r_{ij}}X(ik)(\omega_\nu)\Big)^2$$

$$=\sum_{j=1}^{n}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]^2\Big(\left[\frac{p^*_{ij}}{p_{ij}}\right]D^2X(jk)+E(r^2_{ij})E^2X(jk)\Big)$$

$$+p_{ik}\left[\frac{p_{ik}}{p^*_{ik}}\right]^2E(r^2_{ik})$$

$$D^2X(ik)=\sum_{j=1}^{n}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]D^2X(jk)+\sum_{j=1}^{k}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]^2E(r^2_{ij})\phi^2_j-\phi^2_i$$

or

$$(D^2X(ik))=\Big(I-\Big(p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]\Big)\Big)^{-1}\Big(\sum_{j=1}^{k}p_{ij}\left[\frac{p_{ij}}{p^*_{ij}}\right]^2E(r^2_{ij})\phi^2_j-\phi^2_i\Big)\,.$$

As for $\hat{T}_s$ we have

$$E(\hat{T}_s)=\sum_{i=1}^{k}p_i\left[\frac{p_i}{p^*_i}\right]E\Big(\sum_{\nu=1}^{r_i}X(ik)(\omega_\nu)\Big)$$

$$=\sum_{i=1}^{k} p_i\phi_i = T ,$$

$$E(\hat{T}_s^2)=\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2 E\left(\sum_{\nu=1}^{r_i} X(ik)(\omega_\nu)\right)^2$$

$$=\sum_{i=1}^{n} p_i\left[\frac{p_i}{p_i^*}\right]^2\left(\left[\frac{p_i^*}{p_i}\right]D^2X(ik)+E(r_i^2)\phi_i^2\right)+p_k\left[\frac{p_k}{p_k^*}\right]^2E(r_k^2)$$

$$=\sum_{i=1}^{n} p_i\left[\frac{p_i}{p_i^*}\right]D^2X(ik)+\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2E(r_i^2)\phi_i^2 ,$$

$$D^2(\hat{T}_s)=\left(p_i\left[\frac{p_i}{p_i^*}\right]\right)'\left(I-\left(p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]\right)\right)^{-1}\left(\sum_{j=1}^{k} p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]^2E(r_{ij}^2)\phi_j^2-\phi_i^2\right)$$

$$+\left(\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2E(r_i^2)\phi_i^2-T^2\right) ,$$

$$\sum_{j=1}^{k} p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]^2E(r_{ij}^2)\phi_j^2-\phi_i^2=\sum_{j=1}^{k} p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]^2\left(\sigma_{ij}^2+\left[\frac{p_{ij}^*}{p_{ij}}\right]^2\right)\phi_j^2-\phi_i^2$$

$$=\sum_{j=1}^{k} p_{ij}\left[\frac{p_{ij}}{p_{ij}^*}\right]^2\sigma_{ij}^2\phi_j^2+\sum_{j=1}^{k} p_{ij}(\phi_j-\phi_i)^2$$

$$\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2E(r_i^2)\phi_i^2-T^2=\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2\left(\sigma_i^2+\left[\frac{p_i^*}{p_i}\right]^2\right)\phi_i^2-T^2$$

$$=\sum_{i=1}^{k} p_i\left[\frac{p_i}{p_i^*}\right]^2\sigma_i^2\phi_i^2+\sum_{i=1}^{k} p_i(\phi_i-T)^2.$$

5. Existence of a variable weight process corresponding to a splitting process.

For a splitting process defined by p_is, p_{ij}s, r_is and r_{ij}s with $E(r_i)=\mu_i$, $E(r_{ij})=\mu_{ij}$ where

$$\sum_i p_i\mu_i<1 \qquad \sum_j p_{ij}\mu_{ij}<1$$

$$\mu_i>0 \text{ for } p_i>0 \qquad \mu_{ij}>0 \text{ for } p_{ij}>0$$

hold, we have a variable weight process which, as is easily seen, gives this splitting process as its corresponding model and is defined by

$p_i^*\equiv p_i\mu_i$ for $p_i>0$ $\quad p_{ij}^*\equiv p_{ij}\mu_{ij}$ for $p_{ij}>0$,

$p_{k'}^*\equiv 1-\sum_i p_i\mu_i \quad p_{ik'}^*\equiv 1-\sum_j p_{ij}\mu_{ij}$ for some trap k' other than k

and

$$p_h^* \equiv 0 \qquad p_{ih}^* \equiv 0 \qquad \text{otherwise.}$$

Thus it can be seen that in practical problems with $p_{ik'}$ sufficiently large any splitting process may be considered to be a substitute for some variable weight process.

When we denote by $D^2(\hat{T}_s)$and $D^2(\hat{T}_w)$, variances of estimates for T by splitting process and by its corresponding variable weight process, we have

$$\begin{aligned}\Delta^2(w, s) &\equiv D^2(\hat{T}_s) - D^2(\hat{T}_w) \\ &= (p_i \mu_i^{-1})'(I - (p_{ij}\mu_{ij}^{-1}))^{-1}\left(\sum_{j=1}^{k}(E(r_{ij}^2) - E(r_{ij}))p_{ij}\mu_{ij}^{-2}\phi_j^2\right) \\ &+ \left(\sum_{i=1}^{k}(E(r_i^2) - E(r_i))p_i\mu_i^{-2}\phi_i^2\right).\end{aligned}$$

This formula suggests that in practical cases we should take r_{ij}s as follows: for r_{ij}s with $E(r_{ij}) \leqq 1$ we had better take such r_{ij}s that

$$\text{Prob}\,(r_{ij}=1)=\mu_{ij} \quad \text{Prob}\,(r_{ij}=0)=1-\mu_{ij}\ ,$$

then we should have

$$E(r_{ij}^2) - E(r_{ij}) = 0\ ,$$

and for r_{ij}s with $E(r_{ij}) > 1$
we have

$$E(r_{ij}^2) - E(r_{ij}) = D^2(r_{ij}) + E^2(r_{ij}) - E(r_{ij})$$

so we had better take such r_{ij}s that

$$\text{Prob}\,(r_{ij}=[\mu_{ij}] \text{ or } [\mu_{ij}]+1)=1\ .$$

When we follow the above suggestion, it becomes desirable to take integral μ_{ij}s for $\mu_{ij} > 1$ and to take $r_{ij} \equiv \mu_{ij}$ in practical splitting procedures.

6. Number of random digits required to get one particle history.

By $W_{w,m}(i)$, $W_{s,m}(i)$ we shall represent the total numbers of random digits required for getting the m-th generation of a particle started from the i-th state following the variable weight process and the splitting process respectively.

It can easily be seen that

$$EW_{w,m}(i)=1+\sum_{j=1}^{n}p_{ij}\mu_{ij}EW_{w,m-1}(j)$$

$$EW_{s,m}(i)=1+\sum_{j=1}^{n}p_{ij}\mu_{ij}EW_{s,m-1}(j)\,.$$

In the matrix notation we have

$$\begin{aligned}(EW_{w,m}(i))=(EW_{s,m}(i))&=e+(p_{ij}\mu_{ij})(EW_{w,m-1}(i))\\&=e+(p_{ij}\mu_{ij})(EW_{s,m-1}(i))\\&=(I+(p_{ij}\mu_{ij})+\cdots+(p_{ij}\mu_{ij})^{m-1})e\end{aligned}$$

where $e=(1\cdots 1)'$. Under the condition

$$\sum_{j}p_{ij}\mu_{ij}<1 \qquad i=1, 2, \cdots n$$

we have

$$(EW_{w,m}(i))=(EW_{s,m}(i))\leqq(I-(p_{ij}\mu_{ij}))^{-1}e\,,$$

therefore there exist random variables $W_w(i)$ and $W_s(i)$ $(i=1, 2, \cdots, n)$ such that [3]

$$\begin{aligned}W_{w,m}(i)&\to W_w(i)\\W_{s,m}(i)&\to W_s(i)\,.\end{aligned}\qquad \text{(with probability 1 as } m\to\infty)$$

For W_w and W_s we have

$$(D^2W_w(i))=(I-(p_{ij}\mu_{ij}))^{-1}(2(p_{ij}\mu_{ij})(EW_w(i))+((p_{ij}\mu_{ij})-I)(E^2W_w(i))+e)$$

$$(D^2W_s(i))=(I-(p_{ij}\mu_{ij})^{-1}(2(p_{ij}\mu_{ij})(EW_s(i))+((p_{ij}\overline{r_{ij}^2}-I)(E^2W_s(i))+e)$$

where $\overline{r_{ij}^2}=E(r_{ij}^2)$ and $e=(1, 1, \cdots, 1)'$.

7. Multiplicative splitting process.

When we have a splitting process defined by p_is, p_{ij}s, r_is and r_{ij}s with $E(r_i)=\mu_i$s and $E(r_{ij})=\mu_{ij}$s, we shall call it a multiplicative splitting process if $\mu_i\mu_{ij}=\mu_j$, $\mu_{ij}\mu_{jh}=\mu_{ih}$ hold for $i, j, h=(1, 2, \cdots n, k)$.

For a multiplicative splitting process, we have only to count the total number of particles trapped at the k-th state and we have no need to worry about the weight of each particle, as each particle has one and the same weight μ_k^{-1} when it is trapped at the k-th state.

This simple structure of multiplicative system makes splitting process practicable. Taking into account of $W(i)$s, the existence of $X(ik)$,

$EX(ik)$ and $EX^2(ik)$ is clear, although their existence has been assumed in the previous sections, for multiplicative splitting process and its corresponding variable weight process with μ_{ij}s satisfying the following conditions [3]

$$\sum_j p_{ij}\mu_{ij} < 1 ; \qquad i=1, 2 \cdots , n .$$

When we denote by $X_s(ik)$ and $X_w(ik)$ the total number of particles trapped at the k-th state from a particle started from the i-th state following the splitting process and the corresponding variable weight process respectively, we have

$$(EX_w(ik))=(I-(p_{ij}\mu_{ij}))^{-1}(p_{ik}\mu_{ik})\equiv(\phi_i^*)$$

$$(EX_s(ik))=(\phi_i^*)$$

$$(D^2X_w(ik))=(I-(p_{ij}\mu_{ij}))^{-1}\left(\sum_{j=1}^{k} p_{ij}\mu_{ij}\phi_j^{*2}-\phi_i^{*2}\right)$$

$$(D^2X_s(ik))=(I-(p_{ij}\mu_{ij}))^{-1}\left(\sum_{j=1}^{k} p_{ij}\overline{r_{ij}^2}\phi_j^{*2}-\phi_i^{*2}\right)$$

where $\overline{r_{ij}^2}=E(r_{ij}^2)$.

Now it can be seen that von Neumann's splitting technique has all the desirable properties stated in sections 5 and 7 and thus may be considered to have an optimum character from the practical standpoint.

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