A GENERALIZATION OF LAPLACE CRITERION FOR DECISION PROBLEMS

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(Received Jan. 20, 1956)

0. In this paper, we shall concern ourselves with a generalization of the result of Chernoff [1] on the rational selection of decision problems to the case, in which the set of states of nature constitutes a topological space. As the corollaries to the theorem proved below, we will obtain the results for the cases, which were considered in Uzawa [2].

1. Let the states of nature be a topological space Ω such that

$$\Omega=\bigcup_{\nu=1}^{\infty} B_\nu$$

where B_ν is a compact set for $\nu=1, 2, \cdots$, and

$$B_1 \subset B_2 \subset \cdots ,$$

$\sum=\{\sigma\}$ be a "complete" group of transformations σ on Ω and μ a measure on Ω which is invariant with respect to $\sum$ and finite on B_ν:

$$0<\mu(B_\nu)<\infty \qquad (\nu=1, 2, \cdots) .$$

By a *transformation* on Ω we mean a topological mapping from Ω onto Ω. A set of transformations $\sum=\{\sigma\}$ is a *group*, if $\sum$ contains $\sigma\tau^{-1}$ for any $\sigma, \tau \in \sum$. A group $\sum$ of transformations on Ω will be called to be "*complete*,"—the terminology is used only in this paper—if there is no other open set of Ω than Ω itself and the empty set ϕ, such that

$$A^\sigma=A \qquad \text{for every} \quad \sigma \in \sum .$$

A measure μ on Ω is said to be *invariant* with respect to $\sum$, if, for any measurable set A and any transformation $\sigma \in \sum$, A^σ also is measurable and

$$\mu(A^\sigma)=\mu(A) .$$

A non-empty set X of functions defined on Ω is said to be a *randomized normal problem*, if the following conditions are fulfilled:

(1) $x(t) \geqq 0$ for all $x \in X$ and $t \in \Omega$.

(2) $X \in L(\Omega)$, where $L(\Omega)$ is the set of all bounded and continuous functions defined on Ω.

(3) There exists a finite number K such that

$$\int_{\Omega} x(t)\mu(dt) < K$$

for all $x \in X$.

(4) X is closed, i.e. if $\lim_{\nu\to\infty} x_\nu(t) = x(t)$ $(t \in \Omega)$ and $x_\nu \in X$ $(\nu = 1, 2, \cdots)$, then $x \in X$.

(5) X is convex, i.e. if $x, y \in X$ and $0 < \lambda < 1$, then

$$\lambda \cdot x + (1-\lambda) \cdot y \in X .$$

2. When a set of states of nature Ω and a complete transformation group $\sum$ on Ω are given, a class of randomized normal problems $G = \{X\}$ on Ω will be called a *general problem*, when G has the following properties:

(1) If $X \in G$ and $\alpha > 0$ and y is a non-negative functions on Ω belonging to $L(\Omega)$ such that

$$\int_{\Omega} y(t)\mu(dt) < \infty$$

then the set

$$\alpha X + y = \{\alpha x + y \, ; \, x \in X\}$$

belongs to G.

(2) For any $x_1, \cdots, x_s$ in $L(\Omega)$ such that

$$x_1, \cdots, x_s \geqq 0$$

$$\int_{\Omega} x_1(t)\mu(dt) < \infty, \cdots, \int x_s(t)\mu(dt) < \infty$$

the set $[x_1, \cdots, x_s]$ belongs to G, where

$$[x_1, \cdots, x_s] = \left\{ \sum_{1=1}^{s} \alpha_i x_i \, ; \, \alpha_1, \cdots, \alpha_s \geqq 0, \sum_i \alpha_i \leqq 1 \right\} .$$

(3) For any positive number c_0, the set

$$X_0 = \left\{ x \, ; \, x \in L(\Omega), \, x \geqq 0 \quad \text{and} \quad \int_{\Omega} x(t)\mu(dt) \leqq c_0 \right\}$$

belongs to G.

3. We write, as usual, $x \geqq y$ for functions x, y on Ω, when $x(t) \geqq y(t)$ for any $t \in \Omega$, and $x \geq y$, when $x \geqq y$ and $x \neq y$.

For any set $X = \{x\}$ of functions on Ω, an element x_0 of X is said to be *efficient* in X, if $x \geq x_0$ for no $x \in X$. The set of all efficient elements in X is denoted by $E(X)$.

A selection C, by which each problem X of G is associated with a subset $C(X)$ of X, is said to be *rational*, if it satisfies the following five postulates:

P1. $C(X) \subset E(X)$.

P2. $Y \subset X$ and $X, Y \in G$ imply $C(Y) \supset C(X) \cap Y$.

P3. $C(X)$ is closed and convex for any $X \in G$.

P4. If $Y = \alpha X + y$, where $X \in G$ and $\alpha > 0$ and y is a function in $L(\Omega)$ such that

$$y(t) \geqq 0 \,, \qquad \int_\Omega y(t)\mu(dt) < \infty$$

then

$$C(Y) = \alpha C(X) + y \,.$$

P5. If $X^\sigma = X$ for $X \in G$ and $\sigma \in \sum$, then

$$C(X)^\sigma = C(X) \,.$$

For a set X of functions on Ω, we shall define the set X^σ as follows:

$$X^\sigma = \{x^\sigma \,; x \in X\}$$

where

$$x^\sigma(t^\sigma) = x(t) \qquad (t \in \Omega) \,.$$

4. For any set X of functions on Ω, the subset

$$L(X) = \left\{x_0 \,; x_0 \in X \quad \text{and} \quad \int_\Omega x_0(t)\,\mu(dt) = \max_{x \in X} \int_\Omega x(t)\,\mu(dt) \right\}$$

of X will be called *Laplace set* of X.

THEOREM 1: *The rational selection $C(X)$ for a general problem $G = \{X\}$ for a topological space Ω coincides with Laplace set $L(X)$*:

$$C(X) = L(X) \,.$$

5. PROOF OF THEOREM 1. (1) Let

$$X_0 = \left\{x \,; x \geqq 0,\ x \in L(\Omega) \ \text{and} \ \int_\Omega x(t)\mu(dt) \leqq c_0 \right\}$$

which belongs to G. Then we shall prove that

$$C(X_0) = \left\{x \,; x \geqq 0, x \in L(\Omega) \quad \text{and} \quad \int_\Omega x(t)\mu(dt) = c_0 \right\} .$$

If $x \in X_0$ and $\int_\Omega x(t)\mu(dt)=c<c_0$, then $\frac{c_0}{c}x(t) \in X_0$ and $\frac{c_0}{c}x(t) \geq x(t)$. Therefore,

$$x \notin E(X)$$

and a fortiori

$$x \notin C(X) \quad \text{by P1.}$$

Let x_0 be a function in X_0 such that

$$\int_\Omega x_0(t)\mu(dt)=c_0 .$$

Since $x_0(t)$ is continuous, there exists an open set A so that

$$x_0(t)>0 \quad \text{for all} \quad t \in A .$$

For any function $x(t)$ in X_0 such that

$$\int_\Omega x_0(t)\mu(dt)=c_0$$

there exists a sequence $\{x_\nu\}$ of functions on Ω such that

$$x_\nu \in X_0$$

$$\int_\Omega x_\nu(t)\mu(dt)=c_0$$

and

$$\{t\,;x_\nu(t)>0\} \subset B_\nu \qquad (\nu=1, 2, \cdots) .$$

We shall show that $x_\nu \in C(X_0)$ for any $\nu=1, 2, \cdots$. Since B_ν is compact, there exists a finite set of transformations $\sigma_1, \cdots, \sigma_m$ so that

$$B \subset A^{\sigma_1} \cup \cdots \cup A^{\sigma_m} .$$

Then

$$y=\frac{1}{m}\sum_{j=1}^{m} x^{\sigma_j}$$

is in $C(X_0)$, by virtue of P5, and

$$y(t)>0 \quad \text{for all} \quad t \in B_\nu .$$

Therefore

$$\alpha=\inf_{t \in B_\nu} y(t)>0$$

and

$$0<\beta=\sup_{t \in B_\nu} x_\nu(t)<\infty .$$

Then the function

$$z(t)=\frac{y(t)-\dfrac{\alpha}{\beta}x_{\nu}(t)}{1-\dfrac{\alpha}{\beta}}$$

belongs to X_0 and

$$y(t)=\frac{\alpha}{\beta}x_{\nu}(t)+\left(1-\frac{\alpha}{\beta}\right)z(t) .$$

Since $y\in C(X)$ and $\frac{\alpha}{\beta}>0$, we must have

$$x_{\nu}(t)\in C(X_0) .$$

From P4 we conclude that $x\in C(X_0)$.

(2) For $X\in G$, let

$$c_0=\max_{x\in X}\int_{\Omega}x(t)\mu(dt)$$

which is finite. Then, if $x_0\in X$ and $\int_{\Omega}x_0(t)\mu(dt)=c_0$, x_0 belongs to $C(X)$. This follows from

$$C(X)\supset C(X_0)\cap X ,$$

where

$$X_0=\left\{x\,;\,x\geqq 0,\,x\in L(\Omega)\quad\text{and}\quad\int_{\Omega}x(t)\mu(dt)\leqq c_0\right\} .$$

(3) If, for $x_1\in X$, we have $\int_{\Omega}x_1(t)\mu(dt)=c_1<c_0$, then we shall show that $x_1\notin C(X)$.

Let x_0 be a function in X such that

$$\int_{\Omega}x_0(t)\mu(dt)=c_0 ,$$

and $$Y=[x_0,x_1]=\{\alpha x_0+\beta x_1\,;\,\alpha,\beta\geqq 0\quad\text{and}\quad\alpha+\beta\leqq 1\} .$$

If $x_1\in C(X)$, it can be shown that this leads to a contradiction. Namely in that case, we would have $Y\in G$ and $x_0, x_1\in C(X)$. Therefore

$$\{\lambda x_0+(1-\lambda)x_1\,;\,0<\lambda<1\}\subset C(Y) .$$

Set

$$Z=\left\{z\,;\,z\geqq 0,\,z\in L(\Omega)\quad\text{and}\quad\int_{\Omega}z(t)\mu(dt)\leqq\frac{c_0+c_1}{2}\right\}$$

and $$V=Y\cap Z .$$

Then $x_1 \in C(Y) \cap V$ and $x_2, x_3 \in C(Z) \cap V$,

where

$$x_2 = \frac{1}{2}(x_0 + x_1) \quad \text{and} \quad x_3 = \frac{c_1 + c_0}{2c_0} x_0 .$$

Hence x_1, x_2 and x_3 belong to $C(V)$, and therefore

$$\frac{1}{2}(x_1 + x_3) \in C(V) ,$$

On the other hand,

$$\frac{1}{2}(x_1 + x_3) \leq \frac{1}{2}(x_1 + x_0) = x_2 \in V .$$

Hence

$$\frac{1}{2}(x_1 + x_3) \notin E(V) ,$$

which contradicts to P1, q.e.d.

4. For a topological space Ω, in general, the existence of a complete transformation group $\sum$ or an invariant measure μ on Ω is very dubious. For a *finite set* $\Omega = \{1, \cdots, n\}$, however, the complete transformation group $\sum$ is the symmetric group S_n—the set of all permutations of $\{1, \cdots, n\}$—and the invariant measure μ is the one represented by

$$\mu(1) = \cdots = \mu(n) = \frac{1}{n} .$$

We can, therefore, deduce the following theorem as a special case of theorem 1:

THEOREM 2: *The rational selection $C(X)$ for a general problem $G = \{X\}$ for a finite set $\Omega = \{1, \cdots, n\}$ coincides with the Laplace criterion:*

$$C(X) = \left\{ x_0 ; x_0 \in X \quad \text{and} \quad \frac{1}{n} \sum_{t=1}^{n} x_0(t) = \max_{x \in X} \frac{1}{n} \sum_{t=1}^{n} x(t) \right\} .$$

5. For an *n-dimensional euclidean space* Ω, the group of all *translations* on Ω, i.e.

$$\sum = \{\sigma ; \sigma(t) = t + \sigma (t \in \Omega)\}$$

is complete and the *Lebesgue measure* μ is invariant with respect to Ω. In this case, we have the following theorem:

THEOREM 3: *When the set of states of nature is an n-dimensional euclidean space Ω, the rational selection for a general problem $G=\{X\}$ becomes as follows:*

$$C(X)=\left\{x_0\,;\, x_0\in X \quad \text{and} \quad \int_\Omega x_0(t)\cdot dt = \max_{x\in X} \int_\Omega x(t)dt\right\}\;.$$

REFERENCES

[1] Chernoff, H., "Rational selection of decision problems," *Econometrica*, vol. 22 (1954), pp. 422–43.

[2] Uzawa, H., "Note on rational selection of decision problems," submitted to *Econometrica*.