CENTRAL CONVERGENCE CRITERION IN THE MULTIDIMENSIONAL CASE

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1. Multidimensional central convergence criterion. *If* $S_n = X_{n1} + X_{n2} + \cdots + \cdots + X_{nl_n}$ *are sums of uan p-dimensional independent random vectors and if* b_n *are constant vectors, then* $L(S_n - b_n) \to L$ *with log of ch.f. necessarily of the form*

$$\psi(t) = ia't - \frac{1}{2}t'\sigma t + \int_{R_p}\left(e^{it'x} - 1 - \frac{it'x}{1+|x|^2}\right)d\nu \ , \qquad (t \in R_p) \ , \tag{1}$$

where

$$\int_{|x|<1}|x|^2 d\nu < \infty \ , \qquad \int_{|x|\geqq 1} d\nu < \infty \ ,$$

if, and only if,

(2) *for every continuity set* E *of* ν *such that* $\overline{E} \not\ni 0$,

$$\sum_l \int_E dF_{nl} \to \nu(E) \ ,$$

(3) *as* $n \to \infty$ *and then* $\varepsilon \to 0$,

$$\sum_l \left\{\int_{|x|<\varepsilon} xx'dF_{nl} - \int_{|x|<\varepsilon} xdF_{nl}\int_{|x|<\varepsilon} x'dF_{nl}\right\} \to \sigma \ ,$$

(4) *for a fixed bounded neighborhood* V *of the origin which is a continuity set of* ν

$$\sum_l \int_V xdF_{nl} - b_n \to a + \int_V xd\mu - \int_{V^c}\frac{x}{|x|^2}d\mu \ ,$$

where μ *is the measure defined by*

$$\mu(E) = \int_E \frac{|x|^2}{1+|x|^2}d\nu \ . \tag{5}$$

Under (2), (3) *can be replaced by*

$$\sum_l \left\{\int_{\varepsilon V} xx'dF_{nl} - \int_{\varepsilon V} xdF_{nl}\int_{\varepsilon V} x'dF_{nl}\right\} \to \sigma \ , \text{ as } n \to \infty \text{ and then } \varepsilon \to 0.$$

We use the same notations as in the preceeding paper [4]. If $x_{n,\varepsilon}$

are real numbers (or real matrixes) with double suffixes n, ε ($n=1, 2, \cdots$; $0<\varepsilon<1$), then we write

$$x_{n,\varepsilon} \to a\ , \qquad \text{as } n\to\infty \text{ and then } \varepsilon\to 0,$$

instead of

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} x_{n,\varepsilon}=\lim_{\varepsilon\to 0}\liminf_{n\to\infty} x_{n,\varepsilon}=a\ .$$

The iterated limit in (3) is to be understood in this meaning. We shall denote by $L(a, \sigma, \nu)$ the i.d.l. with log of ch.f. (1).

The method of the proof is essentially the same as used in [2], § 22.4 or [1], § 25 in the one-dimensional case.

PROOF OF THE CRITERION: By the assumption there exist two positive numbers α and β such that

$$S(0, \alpha)\subset V\subset S(0, \beta)\ , \qquad S(0, \alpha)\in \mathfrak{C}_\nu\ ,$$

where $S(0, \rho)$ denotes the open sphere with centre 0 and radius ρ. It is easily proved that under (2), (4) is equivalent to

$$(4')\qquad \sum_l\int_{|x|<\alpha} x dF_{nl}-b_n \to a+\int_{|x|<\alpha} x d\mu-\int_{|x|\geqq\alpha}\frac{x}{|x|^2}d\mu\ .$$

According to [3], § 10, $L(S_n-b_n)\to L(a, \sigma, \nu)$ if, and only if,

$$(6)\qquad \nu_n(E)\to\nu(E)\ , \qquad (\bar{E}\not\ni 0, E\in\mathfrak{C}_\nu)\ ,$$

$$(7)\qquad \int_{R_p}\frac{xx'}{1+|x|^2}d\nu_n\to\sigma+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu\ ,$$

$$(8)\qquad \sum_l\int_{|x|<\alpha} x dF_{nl}+\int_{R_p}\frac{x}{1+|x|^2}d\nu_n-b_n\to a\ ,$$

where ν_n are the measures defined by

$$\nu_n(E)=\sum_l\int_E dF_{nl}(x+a_{nl})\ ,$$

$$a_{nl}=\int_{|x|<\alpha} x dF_{nl}\ .$$

Hence, it suffices to prove that (6), (7) and (8) are equivalent to (2), (3) and (4′).

1° First, we shall prove that (6) and (2) are equivalent to each other. Let E be continuity sets of ν such that $\bar{E}\not\ni 0$.

Put

$$E_n=\bigcap_l \{(E+a_{nl})\cap(E-a_{nl})\}\ , \qquad E'_n=\bigcup_l \{(E+a_{nl})\cup(E-a_{nl})\}\ ,$$

where $E+a=\{x+a\,;\,x\in E\}$, and let $\alpha_n=\max_l \int_{|x|<\alpha} |x|dF_{nl}$ so that, since X_{nl} are uan,

$$\max_l |a_{nl}|\leqq\alpha_n\to 0\ , \tag{9}$$

hence, for every $\varepsilon>0$ and for sufficiently large n

$$\bigcap_{|x|<\varepsilon}(E+x)\subseteq E_n\subseteq E'_n\subseteq\bigcup_{|x|<\varepsilon}(E+x)\ .$$

Since

$$\nu_n(E_n)\leqq\sum_l\int_E dF_{nl}\leqq\nu_n(E'_n)\ ,$$

$$\sum_l\int_{E_n} dF_{nl}\leqq\nu_n(E)\leqq\sum_l\int_{E'_n} dF_{nl}\ ,$$

it follows that (6) and (2) are equivalent, from the following

LEMMA. *Let $\mu_n(n=1, 2, \cdots)$ and μ be measures defined on the family of Borel sets in R_p, and let $A_n(n=1, 2, \cdots)$ and A be Borel sets in R_p. If*

i) *for every continuity set E of μ such that $E\subseteq\bigcup_{|x|<\alpha}(A+x)$ for a given $\alpha>0$,*

$$\mu_n(E)\to\mu(E)\ ,$$

ii) *for every $\varepsilon>0$ there exists an integer N such that*

$$\bigcap_{|x|<\varepsilon}(A+x)\subseteq A_n\subseteq\bigcup_{|x|<\varepsilon}(A+x) \qquad (n\geqq N)\ ,$$

and if

iii) $A\in\mathfrak{C}_\mu$,

then

$$\lim\mu_n(A_n)=\mu(A)\ .$$

PROOF: Put

$$B_\varepsilon=\bigcap_{|x|<\varepsilon}(A+x)\ , \qquad C_\varepsilon=\bigcup_{|x|<\varepsilon}(A+x)\ .$$

Then

$$B_\varepsilon\uparrow A^o \text{ (interior)}, \qquad C_\varepsilon\downarrow\bar{A}\ , \qquad \text{as}\quad \varepsilon\downarrow 0\ ,$$

and if $\varepsilon > \varepsilon' > 0$,

$$\overline{B}_\varepsilon \subseteq B^o_{\varepsilon'} , \qquad C^o_\varepsilon \supseteq \overline{C}_{\varepsilon'} .$$

Hence B^o_ε and $\overline{C}_\varepsilon$ are continuity sets of μ, except for ε belonging to an at most denumerable set. Let B^o_ε and $\overline{C}_\varepsilon$ be continuity sets of μ, so that from (ii) it follows that

$$\mu(B^o_\varepsilon) \leqq \liminf \mu_n(A_n) \leqq \limsup \mu_n(A_n) \leqq \mu(\overline{C}_\varepsilon) .$$

Let $\varepsilon \downarrow 0$, then $\mu(B^o_\varepsilon) \to \mu(A^o) = \mu(A)$ and $\mu(\overline{C}_\varepsilon) \to \mu(\overline{A}) = \mu(A)$. Thus the lemma is proved.

2° We shall prove that under (6) or its equivalent (2), (7) is equivalent to (3). (7) is equivalent to

$$\int_{R_p} \frac{(t'x)^2}{1+|x|^2} d\nu_n \to t'\sigma t + \int_{R_p} \frac{(t'x)^2}{1+|x|^2} d\nu , \qquad (t \in R_p) ,$$

and this is, under (6) or its equivalent (2), equivalent to

$$\int_{|x|<\varepsilon} \frac{(t'x)^2}{1+|x|^2} d\nu_n \to t'\sigma t \quad \text{as } n\to\infty \text{ and then } \varepsilon \to 0, \ (t \in R_p). \tag{10}$$

Since

$$\int_{|x|<\varepsilon} \frac{(t'x)^2}{1+|x|^2} d\nu_n \leqq \int_{|x|<\varepsilon} (t'x)^2 d\nu_n \leqq (1+\varepsilon^2) \int_{|x|<\varepsilon} \frac{(t'x)^2}{1+|x|^2} d\nu_n ,$$

(10) is equivalent to

$$\int_{|x|<\varepsilon} (t'x)^2 d\nu_n \to t'\sigma t \quad \text{as } n\to\infty \text{ and then } \varepsilon\to 0, \ (t \in R_p),$$

and hence to

$$\int_{|x|<\varepsilon} xx' d\nu_n \to \sigma \quad \text{as } n\to\infty \text{ and then } \varepsilon\to 0,$$

which is rewritten as

$$\sum_l \int_{|x|<\varepsilon} xx' dF_{nl}(x+a_{nl}) \to \sigma \quad \text{as } n\to\infty \text{ and then } \varepsilon\to 0. \tag{11}$$

But, on account of (6) and (9), as $n\to\infty$ and then $\varepsilon\to 0$, we have

$$\left| \sum_l \int_{|x|<\varepsilon} x_j x_k dF_{nl}(x+a_{nl}) - \sum_l \int_{|x+a_{nl}|<\varepsilon} x_j x_k dF_{nl}(x+a_{nl}) \right|$$

$$\leqq (\varepsilon+\alpha_n)^2 \sum_l \int_{|x|\geqq \varepsilon-\alpha_n} dF_{nl}(x+a_{nl}) \to 0 , \qquad j=1, 2, \cdots, p,$$

from which it follows that (11) is equivalent to

$$\sum_l \int_{|x|<\varepsilon} (x-a_{nl})(x-a_{nl})' dF_{nl} \to \sigma \qquad \text{as } n\to\infty \text{ and then } \varepsilon\to 0.$$

On the other hand, for $\varepsilon < \alpha$,

$$\sum_l \int_{|x|<\varepsilon} (x-a_{nl})(x-a_{nl})' dF_{nl} - \sum_l \left\{ \int_{|x|<\varepsilon} xx' dF_{nl} - \int_{|x|<\varepsilon} x dF_{nl} \int_{|x|<\varepsilon} x' dF_{nl} \right\}$$

$$= \sum_l \left\{ \int_{\varepsilon \leqq |x| < \alpha} x dF_{nl} \int_{\varepsilon \leqq |x| < \alpha} x' dF_{nl} - a_{nl} a_{nl}' \int_{|x| \geqq \varepsilon} dF_{nl} \right\} ,$$

every element of which is, in the absolute value, less than

$$\left(\alpha^2 \max_l \int_{|x|\geqq\varepsilon} dF_{nl} + \alpha_n^2 \right) \sum_l \int_{|x|\geqq\varepsilon} dF_{nl} \to 0 .$$

Therefore, under (6) or its equivalent (2), (7) is equivalent to (3).

3° It remains to prove that under (6) and (7), or equivalently under (2) and (3), (8) is equivalent to (4′). Define measures μ_n by (5) with ν replaced by ν_n, then from (6) it follows that

$$\mu_n(E) \to \mu(E) , \qquad (\bar{E} \not\ni 0, E \in \mathfrak{C}_\mu) ,$$

and from (7) it follows that

$$\mu_n(R_p) \text{ are bounded.}$$

Since

$$\int_{R_p} \frac{x}{1+|x|^2} d\nu_n = \int_{R_p} \frac{x}{|x|^2} d\mu_n$$

$$= \int_{|x|<\alpha} \frac{1+|x|^2}{|x|^2} x d\mu_n - \int_{|x|<\alpha} x d\mu_n + \int_{|x|\geqq\alpha} \frac{x}{|x|^2} d\mu_n ,$$

and we have, by the extended Helley-Bray lemma (see [4], § 1, E),

$$\int_{|x|<\alpha} x d\mu_n \to \int_{|x|<\alpha} x d\mu ,$$

$$\int_{|x|\geqq\alpha} \frac{x}{|x|^2} d\mu_n \to \int_{|x|\geqq\alpha} \frac{x}{|x|^2} d\mu ,$$

it suffices to prove that

$$\int_{|x|<\alpha} \frac{1+|x|^2}{|x|^2} x d\mu_n \to 0 ,$$

or that

$$\sum_l \int_{|x|<\alpha} x dF_{nl}(x+a_{nl}) \to 0 .$$

This follows from the fact that $\alpha_n \to 0$ and $\{x; |x| < \alpha\}$ is a continuity set of ν, so that, by (2) for every $j=1, 2, \cdots, p$,

$$\left| \sum_l \int_{|x|<\alpha} x_j dF_{nl}(x+a_{nl}) \right|$$

$$\leqq \left| \sum_l \int_{|x|<\alpha} (x_j - a_{nlj}) dF_{nl} \right|$$

$$+ \left| \sum_l \int_{|x-a_{nl}|<\alpha} (x_j - a_{nlj}) dF_{nl} - \sum_l \int_{|x|<\alpha} (x_j - a_{nlj}) dF_{nl} \right|$$

$$\leqq \alpha_n \sum_l \int_{|x|\geq\alpha} dF_{nl} + (\alpha+\alpha_n) \sum_l \int_{\alpha-\alpha_n \leqq |x| < \alpha+\alpha_n} dF_{nl} \to 0 ,$$

where a_{nlj} are the jth components of a_{nl}. Thus the criterion is completely proved.

2. Particular cases: We apply now the central convergence criterion to the following particular cases, which serve for elusidating the meaning of the conditions.

Let $S_n = X_{n1} + X_{n2} + \cdots + X_{nl_n}$ be sums of independent p-dimensional random vectors, and let b_n be constant vectors.

1° Normal convergence. A normal law $N(m, \sigma)$ with mean vector m and covariance matrix σ, corresponds to $\psi(t) = im't - \frac{1}{2}t'\sigma t$ and hence, to $L(m, \sigma, 0)$.

Normal convergence criterion. $L(S_n - b_n) \to N(m, \sigma)$ and X_{nl} are uan if, and only if,

$$\sum_l \int_{|x|\geq\varepsilon} dF_{nl} \to 0 , \qquad (\varepsilon > 0) ,$$

$$\sum_l \left\{ \int_V xx' dF_{nl} - \int_V x dF_{nl} \int_V x' dF_{nl} \right\} \to \sigma ,$$

$$\sum_l \int_V x dF_{nl} - b_n \to m ,$$

where V is an arbitrarily fixed bounded neighborhood of the origin.

2° Poisson convergence. A Poisson law $P(a; \lambda)$ (see [4], §4, 2°) corresponds to $L\left(\frac{\lambda a}{1+|a|^2}, 0, \lambda \chi_a\right)$.

Poisson convergence criterion. If the summands X_{nl} are uan, then $L(S_n - b_n) \to P(a; \lambda)$ if, and only if,

$$\sum_l \int_{|x-a|<\varepsilon} dF_{nl} \to \lambda\,, \qquad (0<\varepsilon<|a|)\,,$$

$$\sum_l \int_{|x|\geqq\varepsilon,\,|x-a|\geqq\varepsilon} dF_{nl} \to 0\,, \qquad (\varepsilon>0)\,,$$

$$\sum_l \left\{\int_V |x|^2 dF_{nl} - \left|\int_V x dF_{nl}\right|^2\right\} \to 0\,,$$

$$\sum_l \int_V x dF_{nl}(x) - b_n \to 0\,,$$

where V is an arbitrarily fixed bounded neighborhood of the origin such that $\overline{V} \not\ni a$.

3° Convergence to the finite convolution of Poisson laws. If the summands X_{nl} are uan, then $L(S_n-b_n) \to P(a_1\,;\lambda_1) * P(a_2\,;\lambda_2) * \cdots * P(a_J;\lambda_J)$ where $a_j \neq a_k$ for $j \neq k$ if, and only if,

$$\sum_l \int_{|x-a_j|<\varepsilon} dF_{nl} \to \lambda_j \qquad (0<\varepsilon<\varepsilon_0\,,\ j=1, 2, \cdots, J)$$

$$\sum_l \int_{|x|\geqq\varepsilon,|x-a_j|\geqq\varepsilon,j=1,2,\ldots,J} dF_{nl} \to 0 \qquad (\varepsilon>0)$$

$$\sum_l \left(\int_V |x|^2 dF_{nl} - \left|\int_V x dF_{nl}\right|^2\right) \to 0$$

$$\sum_l \int_V x dF_{nl} - b_n \to 0$$

where V is an arbitrarily fixed bounded neighborhood of the origin such that $\overline{V} \not\ni a_1, a_2, \cdots, a_J$, and

$$\varepsilon_0 = \min_{0\leqq j<k\leqq J} |a_j - a_k|$$

with $a_0=0$.

4° Convergence to the convolution of a normal law and finite number of Poisson laws. If the summands X_{nl} are uan, then $L(S_n-b_n) \to N(m, \sigma) * P(a_1\,;\lambda_1) * \cdots * P(a_J\,;\lambda_J)$ where $a_j \neq a_k$ for $j \neq k$, if, and only if,

$$\sum_l \int_{|x-a_j|<\varepsilon} dF_{nl} \to \lambda_j\,, \qquad (0<\varepsilon<\varepsilon_0\,,\ j=1, 2, \cdots, J)\,,$$

$$\sum_l \int_{|x|\geqq\varepsilon,|x-a_j|\geqq\varepsilon,j=1,2,\ldots,J} dF_{nl} \to 0\,, \qquad (\varepsilon>0)\,,$$

$$\sum_l \left\{\int_V xx' dF_{nl} - \int_V x dF_{nl} \int_V x' dF_{nl}\right\} \to \sigma\,,$$

$$\sum_l \int_V x dF_{nl} - b_n \to m\,,$$

where V is an arbitrarily fixed bounded neighborhood of the origin such that $\overline{V} \not\ni a_1, a_2, \cdots, a_J$.

3. Cases of normed sums. Let $S_n = X_1 + X_2 + \cdots + X_n$ be consecutive sums of independent random vectors X_l with d.f.'s F_l, and let $D_n(\alpha)$ be α-dispersions of S_n in the sense as defined in [3], § 5. Let us assume that

$$\max_{1 \leqq l \leqq n} \int_{|x| \geqq \varepsilon D_n(\alpha)} dF_l \to 0 , \qquad (\varepsilon > 0) , \tag{12}$$

for some α, and fix such an α. Put $D_n = D_n(\alpha)$. From the central convergence criterion and Theorem 5.6 of [3], we have the following

Theorem. Assume that (12) holds. Then there exists a limit law, with positive α-dispersion, of normed sums $\frac{S_n}{\alpha_n} - b_n$ for suitable scaling and centering constants α_n and b_n if, and only if, there exist a measure ν and a matrix σ such that

$$\sum_{l=1}^{n} \int_{D_n E} dF_l \to \nu(E) , \qquad (\bar{E} \not\ni 0, \; E \in \mathfrak{C}_\nu) ,$$

$$\int_{|x|<1} |x|^2 d\nu < \infty , \qquad \int_{|x| \geqq 1} d\nu < \infty ,$$

$$\frac{1}{D_n^2} \sum_{l=1}^{n} \left\{ \int_{|x|<\varepsilon D_n} x x' dF_l - \int_{|x|<\varepsilon D_n} x dF_l \int_{|x|<\varepsilon D_n} x' dF_l \right\} \to \sigma$$

as $n \to \infty$ and then $\varepsilon \to 0$.

If these conditions hold, then for any constant vector a,

$$L\left(\frac{S_n}{D_n} - b_n\right) \to L(a, \sigma, \nu)$$

with

$$b_n = \frac{1}{D_n} \sum_{l=1}^{n} \int_{D_n V} x dF_l - a - \int_V \frac{|x|^2 x}{1+|x|^2} d\nu + \int_{V^c} \frac{x}{1+|x|^2} d\nu + o(1) ,$$

where V is a bounded neighborhood of the origin which is a continuity set of ν.

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