MULTIDIMENSIONAL CENTRAL LIMIT CRITERION IN THE CASE OF BOUNDED VARIANCES

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Summary and Introduction

The purpose of this paper is to investigate the multidimensional central limit problem (in the sense of M. Loève [3]) in the case of bounded variances. The main results are Theorems 2 and 3 in section 3. Applications to particular cases which serve for elusidating the meaning of the conditions of the criterion are treated in section 4.

The corresponding problem in the case of one-dimensional random variables has been completely solved (see, for instance, B. V. Gnedenko and A. N. Kolmogorov [1], § 21 and M. Loève [3], § 21), but the generalization to the case of multidimensional random vectors has not yet appeared as far as I know.

1. Preliminary lemmas

The family of p-dimensional i.d. (infinitely divisible) ch. f.'s coincides with the family of ch. f.'s of the form e^{ψ} with

$$(1) \qquad \psi(t)=ia't-\frac{1}{2}t'\sigma t+\int_{R_p}\left(e^{it'x}-1-\frac{it'x}{1+x'x}\right)d\nu, \qquad (t\in R_p),$$

where

$a\in R_p$ (p-dimensional Euclidean space of column vectors),
σ is a non-negative definite matrix of the pth order,
ν is a measure defined on the family of Borel sets,

$$\int_{|x|<1}|x|^2 d\nu<\infty, \qquad \int_{|x|\geqq 1}d\nu<\infty, \qquad |x|=\sqrt{x'x}\ ,$$

and usual vector notations are used (see P. Lévy [2], § 62). We shall denote by $L(a, \sigma, \nu)$ the i.d.l. defined by the Lévy's formula (1).

We shall use the following facts.

A. *A sequence of i.d.l.'s $L(a_n, \sigma_n, \nu_n)$ converges to an i.d.l. $L(a, \sigma, \nu)$ as $n\to\infty$ if, and only if,*

$$\nu_n(E)\to\nu(E), \qquad (E\in\mathfrak{C}_\nu, \quad \bar{E}\not\ni 0),$$

$$\sigma_n+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu_n\to\sigma+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu\ ,$$

$$a_n \to a .$$

(see [4], Theorem 8.3)

Here and throughout this paper, $\mathfrak{C}_\nu$ denotes the family of continuity sets of a measure ν; $\bar{E}$ denotes the closure of a set E in the Eulclidean space R_p; limits are considered for $n\to\infty$, if not otherwise stated; and for a proposition $P(x)$ and a condition $C(x)$ containing a variable x,

$$P(x), \qquad (C(x)),$$

means that $P(x)$ holds for every x which satisfies $C(x)$.

If X is a random vector with finite mean vector EX, the variance of X or of $L(X)$ is defined by

$$v(X) = \mathrm{E}|X - \mathrm{E}X|^2 ,$$

where $L(X)$ denotes the probability law of X.

B. *An i.d.l.* $L(a, \sigma, \nu)$ *has the finite variance if and only if,*

$$\int_{R_p} |x|^2 d\nu < \infty .$$

When this holds, consider the measure κ defined by

$$\kappa(E) = \int_E |x|^2 \, d\nu .$$

Then, we have

$$\kappa(R_p) < \infty , \qquad \kappa(\{0\}) = 0,$$

and (1) can be rewritten in the form

$$\psi(t) = i\bar{a}'t - \frac{1}{2} t'\sigma t + \int_{R_p} (e^{it'x} - 1 - it'x) \frac{1}{x'x} d\kappa ,$$

where

$$\bar{a} = a + \int_{R_p} \frac{x}{1+|x|^2} d\kappa .$$

We shall denote by $K(a, \sigma, \kappa)$ the i.d.l. defined by the Kolmogorov's formula

$$\psi(t) = ia't - \frac{1}{2} t'\sigma t + \int_{R_p} (e^{it'x} - 1 - it'x) \frac{1}{x'x} d\kappa$$

where $\quad \kappa(R_p) < \infty , \qquad \kappa(\{0\}) = 0 .$

C. *The mean vector* m *and the covariance matrix (second order central moments matrix)* s *of an i.d.l.* $K(a, \sigma, \kappa)$ *are given by*

$$m=a,$$

$$s=\sigma+\int_{R_p}\frac{xx'}{|x|^2}d\kappa\ ,$$

hence the variance v is given by

$$v=\operatorname{tr}\sigma+\kappa(R_p)\ ,$$

where tr *means 'trace of'.*

D. *Let $L_n(n=1, 2, \cdots)$ be i.d.l.'s with bounded variances. If L_n converge to a law L, necessarilly i.d., then L also has the finite variance.*

PROOF: Write

$$L_n=L(a_n,\ \sigma_n,\ \nu_n),\qquad L=L(a,\ \sigma,\ \nu)\ .$$

From the hypothesis we have

$$(2)\qquad \int_{R_p}|x|^2 d\nu_n\leqq C<\infty\ ,\qquad n=1,\ 2,\cdots,$$

and

$$\nu_n(E)\to\nu(E)\ ,\qquad (E\in\mathfrak{C}_\nu\ ,\ \ \overline{E}\not\ni 0)\ .$$

Throughout this paper, C denotes a constant independent of n. Let ε and α be positive numbers such that $\{x;\ \varepsilon<|x|<\alpha\}\in\mathfrak{C}_\nu$, then

$$\int_{\varepsilon<|x|<\alpha}|x|^2 d\nu=\lim_{n\to\infty}\int_{\varepsilon<|x|<\alpha}|x|^2 d\nu_n\leqq\liminf_{n\to\infty}\int_{R_p}|x|^2 d\nu_n\leqq C\ .$$

Let $\varepsilon\to 0$, $\alpha\to\infty$, then we have

$$\int_{R_p}|x|^2 d\nu\leqq C<\infty\ ,$$

and the proposition is proved by B.

REMARK. If the sequence $L(a_n,\ \sigma_n,\ \nu_n)$ converges, then $\operatorname{tr}\sigma_n$ are bounded, and hence the variances of $L(a_n,\ \sigma_n,\ \nu_n)$ are bounded if and only if (2) holds.

E. *Let $\mu_n(n=1,\ 2,\cdots)$ and μ be p-dimensional measures, A a continuity set of μ with $\mu(A)<\infty$, and let $h(x)$ be a real-valued function which is bounded and continuous on $\overline{A}$.*

1° *If*

$$\mu_n(E)\to\mu(E)\ ,\qquad (E\in\mathfrak{C}_\mu\ ,\ \ E\subseteq A)\ ,$$

then

$$\int_A h(x)d\mu_n\to\int_A h(x)d\mu\ .$$

2° *Let a be a finite point in R_p or ∞. If*

$$\mu_n(E)\to\mu(E)\,,\qquad (E\in\mathfrak{C}_\mu,\ \ \bar{E}\not\ni a\,,\ \ E\subseteq A)\,,$$
$$\mu_n(A)\leqq C<\infty\,,\qquad n=1,\ 2,\cdots\,,$$
$$h(x)\to 0\,,\qquad as\ \ x\to a\,,$$

then

$$\int_A h(x)d\mu_n\to\int_A h(x)\,d\mu\,.$$

(Extended Helly-Bray lemma)
Here, $\bar{E}\not\ni\infty$ and $x\to\infty$ mean respectively that E is bounded, and that $|x|\to+\infty$, that is, $R_p+\infty$ is the one-point compactification of R_p.

2. Convergence theorem

THEOREM 1. *Put $L_n=K(a_n,\sigma_n,\kappa_n)$, $(n=1,2,\cdots)$, and $L=K(a,\sigma,\kappa)$. Assume that*

$$\kappa_n(R_p)\leqq C<\infty\,,\qquad n=1,\ 2,\cdots\,, \tag{3}$$

Then $L_n\to L$ if, and only if,

$$\kappa_n(E)\to\kappa(E)\,.\qquad (E\in\mathfrak{C}_\kappa\,,\ \ \bar{E}\not\ni 0\,,\ \infty) \tag{4}$$

$$\sigma_n+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa_n\to\sigma+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa\,, \tag{5}$$

$$a_n\to a\,, \tag{6}$$

where ε is an arbitrarily fixed positive number such that $\{x;\,|x|<\varepsilon\}\in\mathfrak{C}_\kappa$. If (3) *is strengthened by $v_n\to v$, where v_n and v are variances of L_n and L, that is, by*

$$\mathrm{tr}\,\sigma_n+\kappa_n(R_p)\to\mathrm{tr}\,\sigma+\kappa(R_p)\,, \tag{7}$$

then (4) *and* (5) *can be strengthened by*

$$\kappa_n(E)\to\kappa(E)\,,\qquad (E\in\mathfrak{C}_\kappa\,,\ \ \bar{E}\not\ni 0)\,, \tag{8}$$

$$\sigma_n+\int_{R_p}\frac{xx'}{|x|^2}d\kappa_n\to\sigma+\int_{R_p}\frac{xx'}{|x|^2}d\kappa\,. \tag{9}$$

Note that (6) *and* (9) *mean, respectively, that the means of L_n converge to that of L, and that the covariance matrixes of L_n converge to that of L.*

PROOF: We can write as

$$L=K(a,\ \sigma,\ \kappa)=L(\bar{a},\ \sigma,\ \nu)\,.$$

Then

$$\kappa(E)=\int_E |x|^2 d\nu\ , \qquad \nu(E)=\int_E \frac{1}{|x|^2}d\kappa\ , \qquad (\bar{E}\not\ni 0)\ ,$$

$$\bar{a}=a-\int_{R_p}\frac{x}{1+|x|^2}d\kappa\ .$$

We shall omit the corresponding equalities with suffixes n if no confusion occurs. According to A, $L_n\to L$ if, and only if

$$\nu_n(E)\to\nu(E)\ , \qquad (E\in\mathfrak{C}_\nu\ ,\ \ \bar{E}\not\ni 0)\ , \tag{10}$$

$$\sigma_n+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu_n\to\sigma+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu\ , \tag{11}$$

$$\bar{a}_n\to\bar{a}\ . \tag{12}$$

We shall show that (4)–(6) sre equivalent to (10)–(12), by using E repeatedly.

First, (4) and (10) are equivalent under (3). Next,

$$\sigma+\int_{R_p}\frac{xx'}{1+|x|^2}d\nu=\sigma+\int_{R_p}\frac{xx'}{|x|^2(1+|x|^2)}d\kappa$$

$$=\sigma+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa-\int_{|x|<\varepsilon}\frac{xx'}{1+|x|^2}d\kappa+\int_{|x|\geqq\varepsilon}\frac{xx'}{|x|^2(1+|x|^2)}d\kappa\ .$$

Under (4) or its equivalent (10), we have

$$\int_{|x|<\varepsilon}\frac{xx'}{1+|x|^2}d\kappa_n\to\int_{|x|<\varepsilon}\frac{xx'}{1+|x|^2}d\kappa\ ,$$

$$\int_{|x|\geqq\varepsilon}\frac{xx'}{|x|^2(1+|x|^2)}d\kappa_n\to\int_{|x|\geqq\varepsilon}\frac{xx'}{|x|^2(1+|x|^2)}d\kappa\ ,$$

hence, (11) is equivalent to (5). Lastly, under (4) or its equivalent (10), we have

$$\int_{R_p}\frac{x}{1+|x|^2}d\kappa_n\to\int_{R_p}\frac{x}{1+|x|^2}d\kappa\ ,$$

which implies the equivalence of (6) and (12). Thus the first part of the theorem is proved.

It is clear that (5) can be replaced by

$$\sigma_n+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa_n\to\sigma+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa\ , \qquad (\varepsilon\in\mathfrak{C})\ , \tag{5'}$$

where $\mathfrak{C}$ is the set of ε such that $\{x;\ |x|<\varepsilon\}\in\mathfrak{C}_\kappa$. To prove the second part, it suffices to derive (8) and (9) from (4), (5′) and (7). Now, (5′) implies that

$$\operatorname{tr}\sigma_n+\int_{|x|<\varepsilon}d\kappa_n\to\operatorname{tr}\sigma+\int_{|x|<\varepsilon}d\kappa\ ,\qquad (\varepsilon\in\mathfrak{C})\ ,$$

which together with (7) implies

$$\int_{|x|\geqq\varepsilon}d\kappa_n\to\int_{|x|\geqq\varepsilon}d\kappa\ ,\qquad (\varepsilon\in\mathfrak{C})\ .$$

Since $\kappa(R_p)<\infty$, it follows that $\int_{|x|\geqq\varepsilon}d\kappa_n$ and $\int_{|x|\geqq\varepsilon}d\kappa$ converge uniformly to 0 as $\varepsilon\to\infty$, and this together with (4) implies (8). Lastly, (8) and (5) imply (9). Thus, the assertion is completely proved.

COROLLARY 1. *Let $X_n (n=1, 2, \cdots)$ be random vectors with i.d.l.'s and with bounded variances. Then $L(X_n)\to L(X)$ necessarilly an i.d.l. with finite variance if, and only if, $L(X_n-\mathrm{E}X_n)\to L(X-\mathrm{E}X)$ and $\mathrm{E}X_n\to\mathrm{E}X$. If $v(X_n)\leqq C<\infty$ is replaced by $v(X_n)\to v(X)$, then $L(X_n)\to L(X)$ implies that the covariance matrixes of X_n converge to that of X.*

REMARK. Under (4), (5) can be replaced by

$$(13)\qquad \lim_{\varepsilon\to0}\limsup_n\left(\sigma_n+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa_n\right)=\lim_{\varepsilon\to0}\liminf_n\left(\sigma_n+\int_{|x|<\varepsilon}\frac{xx'}{|x|^2}d\kappa_n\right)=\sigma\ .$$

PROOF: It is sufficient to prove that (5′) is equivalent to (13) under (4). Now, (5′) and (13) are, respectively, equivalent to the following conditions

$$(5'')\qquad t'\sigma_n t+\int_{|x|<\varepsilon}\frac{(t'x)^2}{|x|^2}d\kappa_n\to t'\sigma t+\int_{|x|<\varepsilon}\frac{(t'x)^2}{|x|^2}d\kappa\ ,\qquad (\varepsilon\in\mathfrak{C},\quad t\in R_p)\ ,$$

and

$$(13')\qquad \lim_{\varepsilon\to0}\limsup_n\left(t'\sigma_n t+\int_{|x|<\varepsilon}\frac{(t'x)^2}{|x|^2}d\kappa_n\right)$$
$$=\lim_{\varepsilon\to0}\liminf_n\left(t'\sigma_n t+\int_{|x|<\varepsilon}\frac{(t'x)^2}{|x|^2}d\kappa_n\right)=t'\sigma t\ ,\qquad (t\in R_p)\ ,$$

while, (5″) and (13′) are easily seen to be equivalent to one another under (4). Then the remark is proved.

In the next section we shall use the following

COROLLARY 2. *Let $K(a_n, \sigma_n, \kappa_n)$ be a sequence of i.d.l.'s with bounded variances. Then $K(a_n, \sigma_n, \kappa_n)\to L$ necessarily of the form $K(a, \sigma, \kappa)$ if, and only if,*

$$\int_E \frac{1}{|x|^2} d\kappa_n \to \int_E \frac{1}{|x|^2} d\kappa\,, \qquad (E \in \mathfrak{C}_\kappa\,,\quad \bar{E} \not\ni 0)\,,$$

$$\lim_{\varepsilon\to 0} \limsup_n \left(\sigma_n + \int_{|x|<\varepsilon} \frac{xx'}{|x|^2} d\kappa_n\right) = \lim_{\varepsilon\to 0} \liminf_n \left(\sigma_n + \int_{|x|<\varepsilon} \frac{xx'}{|x|^2} d\kappa_n\right) = \sigma\,,$$

$$a_n \to a\,.$$

3. Central convergence criterion

We shall consider a sequence of sums

$$S_n = X_{n1} + X_{n2} + \cdots + X_{nl_n}$$

of independent p-dimensional random vectors X_{nl}, with d.f.'s F_{nl}, subject to the conditions:

(α) $\qquad \mathrm{E}X_{nl} = 0\,,$

(β) The summands X_{nl} are uniformly asymptotically negligible (uan), that is,

$$\max_l P(|X_{nl}| \geqq \varepsilon) \to 0\,, \qquad (\varepsilon > 0)\,.$$

(γ) The X_{nl} have finite variances and

$$v(S_n) \equiv \sum_l v(X_{nl}) \leqq C < \infty\,,$$

where C is a constant independent of n, and $v(X)$ denotes the variance of X.

Comparison lemma. *Let S_n' be random vectors with i.d.l.'s $K(0, 0, \kappa_n)$, where κ_n is the measure defined by*

$$\kappa_n(E) = \sum_l \int_E |x|^2 dF_{nl}(x)\,,$$

then

$$\mathrm{E}S_n' = \mathrm{E}S_n\,, \qquad \sigma(S_n') = \sigma(S_n)\,,$$

where $\sigma(S)$ denotes the covariance matrix of S, and, moreover,

(14) $$\lim L(S_n) = \lim L(S_n')$$

in the sense that, if one of these limit laws exists, so does the other, and both are identical.

PROOF: According to C,

$$\mathrm{E}S_n' = 0 = \mathrm{E}S_n\,,$$

$$\sigma(S_n') = \int_{R_p} \frac{xx'}{|x|^2} d\kappa_n = \sum_l \int_{R_p} xx' dF_{nl}(x) = \sigma(S_n).$$

Let $\psi_n(t)$ be the logarithm of the ch. f.'s of $K(0, 0, \kappa_n)$, then

$$\psi_n(t)=\int_{R_p}(e^{it'x}-1-it'x)\frac{1}{|x|^2}d\kappa_n$$
$$=\sum_l\int_{R_p}(e^{it'x}-1-it'x)dF_{nl}(x)=\sum_l\int_{R_p}(e^{it'x}-1)dF_{nl}(x) .$$

Let $f_n(t)$ be the ch. f.'s of S_n. Then it is proved, in the same way as in the one-dimensional case (see [1], § 21, Theorem 1), that

$$f_n(t)-e^{\psi_n(t)}\to 0 ,$$

from which (14) follows.

THEOREM 2. *If* $S_n=X_{n1}+X_{n2}+\cdots+X_{nl_n}$ *are sums of p-dimensional independent random vectors with mean vectors* 0 *and with finite variances and if*

$$X_{nl} \text{ are uan} ,$$

and

$$v(S_n)\equiv\sum_l v(X_{nl})\leqq C<\infty ,$$

then

1° *the family of limit laws of sequences* $L(S_n)$ *coincides with the family of* $K(0, \sigma, \kappa)$.

2° $L(S_n)\to L(S)$ *necessarily of the form* $K(0, \sigma, \kappa)$ *if, and only if,*

$$\sum_l\int_E dF_{nl}(x)\to\int_E\frac{1}{|x|^2}d\kappa , \qquad (E\in\mathfrak{C}\kappa , \quad \bar{E}\not\ni 0) , \tag{15}$$

$$\lim_{\varepsilon\to 0}\limsup_n\sum_l\int_{|x|<\varepsilon}xx'dF_{nl}(x)=\lim_{\varepsilon\to 0}\liminf_n\sum_l\int_{|x|<\varepsilon}xx'dF_{nl}(x)=\sigma . \tag{16}$$

(15) *can be weakened by*

$$\sum_l\int_E dF_{nl}(x)\to\int_E\frac{1}{|x|^2}d\kappa , \qquad (E\in\mathfrak{C}\kappa , \quad \bar{E}\not\ni 0 , \ \infty) . \tag{17}$$

3° *If* $v(S_n)\leqq C<\infty$ *is strengthened by* $v(S_n)\to v(S)$, *then* (15) *can be strengthened by*

$$\sum_l\int_E |x|^2dF_{nl}(x)\to\kappa(E) , \qquad (E\in\mathfrak{C}\kappa , \quad \bar{E}\not\ni 0) , \tag{18}$$

and this together with (16) *implies*

$$\sum_l\int_{R_p}xx'dF_{nl}(x)\to\sigma+\int_{R_p}\frac{xx'}{|x|^2}d\kappa , \tag{19}$$

which means that the covariance matrixes of S_n *converge to that of* S.

PROOF: Ad 1°. If $L(S_n)\to L(S)$, then $L(S_n')\to L(S)$ by the comparison lemma, and hence $L(S)$ must be of the form $K(0, \sigma, \kappa)$ by Corrollary 1 to Theorem 1. Conversely, let $K(0, \sigma, \kappa)$ be given and let $X_{nl}(l=1, 2, \cdots, n)$ be independent random vectors with the common law $K(0, \sigma/n, \kappa/n)$. Then it is easily seen that X_{nl} satisfy the conditions (α), (β) and (γ) and $L(\sum_{l=1}^{n} X_{nl})=K(0, \sigma, \kappa)\to K(0, \sigma, \kappa)$.

The main part of 2° (the 'if' and 'only if' part) follows from the comparison lemma and Corollary 2 to Theorem 1. The remaining part is proved as follows: (17) implies

$$\kappa_n(E)=\sum_l\int_E |x|^2 dF_{nl}(x)\to\kappa(E) \qquad (E\in\mathfrak{C}\kappa\,,\quad \bar{E}\not\ni 0,\quad \infty)$$

and this together with $\kappa_n(R_p)\leqq C<\infty$ implies (15).

3° follows from the comparison lemma and Theorem 1.

So far the random vectors under consideration had all mean vectors 0. If we suppress this condition, Theorem 2 becomes

THEOREM 3. *If $S_n=X_{n1}+X_{n2}+\cdots X_{nl_n}$ are sums of p-dimensional independent random vectors with finite variances, and if*

$$X_{nl}-\mathrm{E}X_{nl} \text{ are uan,}$$

and

$$v(S_n)\equiv\sum_l v(X_{nl})\leqq C<\infty\,,$$

then $L(S_n)\to L(S)$ necessarily of the form $K(a, \sigma, \kappa)$ if, and only if,

$$\sum_l \mathrm{E}X_{nl}\to a\,,$$

$$\sum_l\int_E dF_{nl}(x+\mathrm{E}X_{nl})\to\int_E\frac{1}{|x|^2}d\kappa\,; \qquad (E\in\mathfrak{C}\kappa\,,\quad \bar{E}\not\ni 0) \tag{20}$$

$$\lim_{\varepsilon\to 0}\limsup_n\sum_l\int_{|x|<\varepsilon} xx'dF_{nl}(x+\mathrm{E}X_{nl}) \tag{21}$$

$$=\lim_{\varepsilon\to 0}\liminf_n\sum_l\int_{|x|<\varepsilon} xx'dF_{nl}(x+\mathrm{E}X_{nl})=\sigma\,.$$

(20) *can be weakened by*

$$\sum_l\int_E dF_{nl}(x+\mathrm{E}X_{nl})\to\int_E\frac{1}{|x|^2}d\kappa\,, \qquad (E\in\mathfrak{C}\kappa\,,\quad \bar{E}\not\ni 0\,,\quad \infty)\,.$$

If $v(S_n)\leqq C<\infty$ is strengthened by $v(S_n)\to v(S)$, then (20) *and* (21) *can be strengthened by*

$$\sum_l \int_E |x|^2 dF_{nl}(x+\mathrm{E}X_{nl}) \to \kappa(E), \qquad (E \in \mathfrak{C}\kappa, \quad \bar{E} \not\ni 0),$$

$$\sum_l \int_{R_p} xx' dF_{nl}(x+\mathrm{E}X_{nl}) \to \sigma + \int_{R_p} \frac{xx'}{|x|^2} d\kappa .$$

4. Particular cases

1° Normal convergence. A normal law $N(0, \sigma)$ with mean vector 0 and covariance matrix σ corresponds to $\psi(t) = -\frac{1}{2} t'\sigma t$ and, hence, to $\kappa(0, \sigma, 0)$. Theorem 2 yields the following

Normal convergence criterion. Let the independent summands X_{nl} be such that $\mathrm{E}X_{nl}=0$, $\sum_l v(X_{nl}) \leqq C < \infty$ for all n, and put $S_n = \sum X_{nl}$. Then $L(S_n) \to$ a normal law $N(0, \sigma)$ and X_{nl} are uan if, and only if,

$$\sum_l \int_{|x| \geqq \varepsilon} dF_{nl}(x) \to 0, \qquad (\varepsilon > 0), \tag{22}$$

$$\sum_l \int_{|x| < \varepsilon} xx' dF_{nl}(x) \to \sigma, \qquad (\varepsilon > 0). \tag{23}$$

And in this case, we have

$$v \leqq \liminf_n v(S_n), \tag{24}$$

where v denotes the variance of the limit law $N(0, \sigma)$. If $v(S_n) \leqq C < \infty$ is replaced by $v(S_n) \to v$, then (22) follows from (23).

We shall show an example in which the inequality (24) strictly holds. Let $N(0, \sigma)$ be a normal law with mean 0 and with variance 1, and let X_{nl}, $l=1, 2, \cdots, n+1$ be independent random vectors such that

i) X_{nl}, $l=1, 2, \cdots, n$ have the common normal law $N\left(0, \frac{\sigma}{2n}\right)$,

ii) $X_{n,n+1} \to 0$ in probability, $\mathrm{E}X_{n,n+1}=0$, and $v(X_{n,n+1}) = \frac{1}{2}$ for all n.

(For example, let $X_{n,n+1}$ be such that $P(X_{n,n+1} = (\pm n, 0, 0, \cdots 0)) = \frac{1}{4n^2}$, $P(X_{n,n+1} = (0, \cdots, 0)) = 1 - \frac{1}{2n^2}$)

Then

$$L\left(\sum_1^n X_{nl}\right) = N\left(0, \frac{\sigma}{2}\right), \qquad X_{n,n+1} \to 0 \qquad \text{in probability},$$

hence,

$$L\Big(\sum_1^{n+1} X_{nl}\Big)\to N\Big(0,\ \frac{\sigma}{2}\Big) \qquad \text{with variance } \frac{1}{2}\,.$$

while

$$v\Big(\sum_1^{n+1} X_{nl}\Big)=v\Big(\sum_1^{n} X_{nl}\Big)+v(X_{n,n+1})=\frac{1}{2}+\frac{1}{2}=1\,.$$

2° Poisson convergence. Let $a(\neq 0)$ be a constant vector $\in R_p$ and let X be a (one-dimensional) Poisson random variable with mean λ, then aX is a p-dimensional random vector and its law $L(aX)$ corresponds to

$$\psi(t)=\lambda(e^{it'a}-1)=i\lambda a't+\lambda(e^{it'a}-1-it'a)\,,$$

and, hence, to $K(\lambda a,\ 0,\ \lambda|a|^2\chi_a)$ where χ_a is the measure defined by

$$\chi_a(E)=\begin{cases}1, & \text{if} \quad E\ni a\,,\\ 0, & \text{if} \quad E\not\ni a\,.\end{cases}$$

We shall denote this p-dimensional probability law by $P(a;\ \lambda)$. Theorem 3 yields the following

Poisson convergence criterion. If the independent summands X_{nl} are such that $X_{nl}-\mathrm{E}X_{nl}$ are uan and $\sum_l v(X_{nl})\leq C<\infty$, for all n, then $L(\sum_l X_{nl})\to P(a;\ \lambda)$ if, and only if,

(25) $$\sum_l \mathrm{E}X_{nl}\to\lambda a$$

(26) $$\sum_l\int_{|x-a|<\varepsilon} dF_{nl}(x+\mathrm{E}X_{nl})\to\lambda\,, \qquad (0<\varepsilon<|a|)\,,$$

(27) $$\sum_l\int_{|x|\geqq\varepsilon,\ |x-a|\geqq\varepsilon} dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,, \qquad (\varepsilon>0)\,,$$

(28) $$\sum_l\int_{|x|<\varepsilon} |x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,, \qquad (\varepsilon<|a|)\,.$$

(27) and (28) can be replaced by

(29) $$\sum_l\int_{|x-a|\geqq\varepsilon,\ |x|<\alpha} |x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,, \qquad (\varepsilon>0,\ \ \alpha>0)\,.$$

If $\sum v(X_{nl})\leq C<\infty$ is replaced by $\sum v(X_{nl})\to\lambda|a|^2$ (variance of the limit law), then (29) is to be replaced by

(30) $$\sum_l\int_{|x-a|\geqq\varepsilon} |x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,, \qquad (\varepsilon>0)\,,$$

which is equivalent to (26), hence, $L(\sum_l X_{nl})\to P(a:\ \lambda)$ if, and only if (25) and (26) hold.

3° Convergence to the finite convolution of Poisson laws. Let $L=P(a_1;\lambda_1)*P(a_2;\lambda_2)*\cdots *P(a_J;\lambda_J)$ be the convolution of $J(=2,3,\cdots)$ Poisson laws, where $a_j\neq a_k$ for $j\neq k$, then

$$L=K(\sum_j \lambda_j a_j,\ 0,\ \sum_j \lambda_j\,|a_j|^2\chi_{a_j}).$$

Put

$$\varepsilon_0=\min_{0\leqq j<k\leqq J}|a_j-a_k|$$

where $a_0=0$. Theorem 3 yields the following

Generalized Poisson convergence criterion. If the independent summands X_{nl} are such that $X_{nl}-\mathrm{E}X_{nl}$ are uan and $\sum_l v(X_{nl})\leqq C<\infty$, then $L(\sum_l X_{nl})\to L=P(a_1;\lambda_1)*P(a_2;\lambda_2)*\cdots *P(a_J,\lambda_J)$ if, and only if,

$$\sum_l \mathrm{E}X_{nl}\to\sum\lambda_j a_j\,, \tag{31}$$

$$\sum_l\int_{|x-a_j|<\varepsilon} dF_{nl}(x+\mathrm{E}X_{nl})\to\lambda_j\,,\quad (0<\varepsilon<\varepsilon_0\,,\ j=1,2,\cdots,J), \tag{32}$$

$$\sum_l\int_{|x|\geqq\varepsilon,\ |x-a_j|\geqq\varepsilon,\ j=1,2,\cdots,J} dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,,\qquad (\varepsilon>0)\,, \tag{33}$$

$$\sum_l\int_{|x|<\varepsilon}|x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,,\qquad (0<\varepsilon<\varepsilon_0)\,. \tag{34}$$

(33) and (34) can be replaced by

$$\sum_l\int_{|x-a_j|\geqq\varepsilon,\ j=1,2,\cdots,J,\ |x|<\alpha}|x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,,\quad (\varepsilon>0\,,\ \ \alpha>0)\,. \tag{35}$$

If $\sum v(X_{nl})\leqq C<\infty$ is replaced by $\sum_l v(X_{nl})\to\sum_j\lambda_j|a_j|^2$ (variance of L) then (35) is to be replaced by

$$\sum_l\int_{|x-a_j|\geqq\varepsilon,\ j=1,2,\cdots,J}|x|^2dF_{nl}(x+\mathrm{E}X_{nl})\to 0\,,\quad (\varepsilon>0)\,, \tag{36}$$

which follows in this case from (32), hence, $L(\sum_l X_{nl})\to L$ if, and only if, (31) and (32) hold.

4° Convergence to the convolution of a normal law and finite number of Poisson laws. Let $L=N(m,\sigma)*P(a_1;\lambda_1)*P(a_2;\lambda_2)*\cdots * P(a_J,\lambda_J)$ be the convolution of a normal law and J Poisson laws, where $a_j\neq a_k$ for $j\neq k$, then

$$L=K(m+\sum\lambda_j a_j,\ \sigma,\ \sum\lambda_j|a_j|^2\chi_{a_j})\,.$$

Convergence criterion. If the independent summands X_{nl} are such that $X_{nl}-\mathrm{E}X_{nl}$ are uan and $\sum v(X_{nl})\leqq C<\infty$ then $L(\sum X_{nl})\to L$

$=N(m, \sigma,) * P(a_1; \lambda_1) * P(a_2; \lambda_2) * \cdots * P(a_J, \lambda_J)$ if, and only if,

(37) $$\sum_l \mathrm{E}X_{nl} \to m + \sum_j \lambda_j a_j ,$$

(38) $$\sum_l \int_{|x-a_j|<\varepsilon} dF_{nl}(x+\mathrm{E}X_{nl}) \to \lambda_j , \qquad (0<\varepsilon<\varepsilon_0,\ j=1, 2, \cdots, J) ,$$

(39) $$\sum_l \int_{|x|\geqq\varepsilon,\ |x-a_j|\geqq\varepsilon,\ j=1,2,\cdots,J} dF_{nl}(x+\mathrm{E}X_{nl}) \to 0 , \qquad (\varepsilon>0) ,$$

(40) $$\sum_l \int_{|x|<\varepsilon} xx' dF_{nl}(x+EX_{nl}) \to \sigma , \qquad (0<\varepsilon<\varepsilon_0) .$$

If $\sum v(X_{nl}) \leqq C < \infty$ is replaced by $\sum v(X_{nl}) \to \mathrm{tr}\, \sigma + \sum_j \lambda_j |a_j|^2$ (variance of L), then (39) follows from (38) and (40).

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