THE SIGNIFICANCE OF THE DISCORDANT VARIANCE ESTIMATES

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1. Summary and introduction. Among the various criteria that have been advanced for testing the homogeneity of a set of variance estimates, M. S. Bartlett's test [1] is generally considered to be the best one available of omnibus variety. However, as C. Eisenhart and H. Solomon [2] have pointed out, 'Bartlett's test, though applicable, is somewhat inappropriate when the alternative to homogeneity that is most likely to represent the true situation, or that is of special importance when it does obtain, is that the expected value of some one of the variance estimates substantially exceeds the expected values of the others, which may be homogeneous or not.'

W. G. Cochran [3] has developed a test for such a situation, that is, a test for the statistical significance of the largest of a set of variance estimates, and if $S_1^2, S_2^2, \dots, S_k^2$ are k variance estimates distributed independently as $\chi^2\sigma^2/n$ with n degrees of freedom, the proposed statistic for testing the significance is

$$(1) \qquad G(k, 1)=\frac{\text{largest } S^2}{S_1^2+S_2^2+\cdots+S_k^2}$$

In practice, however, we are also often confronted with the following situations as regards discordant variance estimates:

(a) The reverse situation to the Cochran's, that is, the situation that the smallest of a set of variance estimates may have the appearance of being discordant.

(b) The situation that the two largest variance estimates may appear to be different from the remaining estimates.

(c) The situation, which is reverse to (b), that the two smallest estimates may appear to be discordant.

Here, we are interested in determining whether these variance estimates having the appearance as mentioned above should be truly regarded as being inconsistent with the remaining variance estimates, in other words, we want to test the hypothesis that all variance estimates,

$S_1^2, S_2^2, \cdots, S_k^2$ are estimates of the same population variance, σ^2, against the alternative hypothesis that the largest (or smallest) or the two largest (or two smallest) variance estimates are estimates of the population variances different from σ^2, whereas the remaining variance estimates have the common variance σ^2 as the expected value.

For testing the significance of the smallest of k variance estimates, $S_1^2, S_2^2, \cdots, S_k^2$, we propose the statistic

$$S(k, 1) = \frac{\text{smallest } S^2}{S_1^2 + S_2^2 + \cdots + S_k^2} \tag{2}$$

and for testing whether the two largest variance estimates are too high, the statistic

$$G(k, 2) = \frac{\sum \text{two largest } S^2\text{s}}{S_1^2 + S_2^2 + \cdots + S_k^2} \tag{3}$$

can be used on the intuitive ground. Similarly we can use the statistic

$$S(k, 2) = \frac{\sum \text{two smallest } S^2\text{s}}{S_1^2 + S_2^2 + \cdots + S_k^2} \tag{4}$$

for the situation (c). In general, we can consider the analogous statistics for testing the significance of the three, four, etc. discordant variance estimates.

In this paper we shall obtain the sampling distribution functions of the statistics $G(k, \nu)$ and $S(k, \nu)$ in the null case when all k variance estimates are homogeneous and we shall also give the favourable expanded forms of the distribution functions to obtain the approximate 5% and 1% points of $G(k, \nu)$ and $S(k, \nu)$. Although we have not investigated the power of the tests in this paper, it is believed that our test procedure possesses considerable intuitive appeal for the practical situations considered in (a), (b) and (c).

In Section 2, we shall discuss the relations between our proposed statistics and the optimum slippage test for variance estimates [4].

2. The relation with the optimum slippage test. Let $S_1^2, S_2^2, \cdots, S_k^2$ be k variance estimates on the basis of samples of the same size drawn from k normal populations $\pi_1, \pi_2, \cdots, \pi_k$, respectively and let $\sigma_i^2 (i=1,2,\cdots,k)$ be the variance of the i th population π_i. If $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_{i-1}^2 = \sigma_{i+1}^2 = \cdots = \sigma_k^2$ and $\sigma_i^2 = \lambda^2 \sigma_1^2$ with $\lambda > 1$, it will be called that π_i has been slipped to the right. Similarly we have the population slipped to the left when $\sigma_i^2 = \mu^2 \sigma_1^2$ with $0 < \mu < 1$.

Let D_0 be the decision that all k variances are equal and $D_i (i=1, 2 \cdots\cdots, k)$ the decision that D_0 is false and π_i has been slipped to the right. D. R. Truax [4] has shown that the optimum statistical procedure, i.e. the procedure for selecting one of the $(k+1)$ decisions $D_0, D_1, \cdots\cdots, D_k$ which, under some restrictions, maximizes the probability of making the correct decision when one of the population has been slipped to the right is the following:

$$\text{if } S_M^2 / \sum_{i=1}^k S_i^2 > L_\alpha, \text{ select } D_M \text{ and}$$
$$\text{if } S_M^2 / \sum_{i=1}^k S_i^2 \geqq L_\alpha, \text{ select } D_0,$$

where M denotes the population yielding the largest variance estimate and L_α is the 100α percentage point of the statistic $S_M^2 / \sum_{i=1}^k S_i^2$. This statistic is the one suggested by Cochran, i.e. $G(k, 1)$. Similarly it is easily seen that the optimum procedure for selecting one of the $(k+1)$ decisions $D_0, D_1', \cdots\cdots, D_k'$ in the case when one of the populations has been slipped to the left is as follows:

$$\text{if } S_N^2 / \sum_{i=1}^k S_i^2 \geqq l_\alpha, \text{ select } D_0 \text{ and}$$
$$\text{if } S_N^2 / \sum_{i=1}^k S_i^2 < l_\alpha, \text{ select } D'_N,$$

where D_i' is the decision that $\sigma_1^2 = \cdots\cdots = \sigma_{i-1}^2 = \sigma_{i+1}^2 = \cdots\cdots = \sigma_k^2 (\equiv \sigma^2)$ and $\sigma_i^2 = \mu^2 \sigma^2$ $(0 < \mu < 1)$ and N denotes the population yielding the smallest variance estimate and l_α is the 100α percentage point of $S(k, 1) = S_N^2 / \sum_{i=1}^k S_i^2$.

In the case when two or more populations have been slipped to the right or to the left, it is, in general, difficult to obtain the optimum statistical procedure such as the one discussed above. But in the special case where the variances of the slipped populations have the high values or have the low values of the same magnitude, we can show that, under the analogous restrictions of Truax's, the statistical procedure based on our statistics $G(k, \nu)$ or $S(k, \nu)$ is optimum.

Let D_{00} be the decision that all k variances are equal and D_{ij} the decision that $\sigma_1^2 = \cdots\cdots = \sigma_{i-1}^2 = \sigma_{i+1}^2 = \cdots\cdots = \sigma_{j-1}^2 = \sigma_{j+1}^2 = \cdots\cdots = \sigma_k^2 (\equiv \sigma^2)$ and $\sigma_i^2 = \sigma_j^2 = \lambda^2 \sigma^2$, $(i < j,\ i = 1, 2, \cdots\cdots, k-1)$, where $\lambda > 1$. Then we have $D_{ij} = D_{ji}$, and the number of the distinct decisions is $\frac{k(k-1)}{2} + 1$. We shall consider the optimum procedure of selecting one of the $\frac{k(k-1)}{2} + 1$ decisions, $D_{00}, D_{12}, \cdots\cdots, D_{k-1,k}$, under the following restrictions:

(i) D_{00} should be selected with probability $1-\alpha$, where α is a small positive number, when all the variances are equal.

(ii) When two populations π_i and π_j have been slipped to the right in the same magnitude, the probability of selecting D_{ij} should be the same for all possible pairs of i and j and

(iii) The procedure should be invariant if all the observations are multiplied by the same positive constant and if some constant b_i is added to all observations in the i th population.

Because of the restriction (iii), any allowable procedure depends only on $(k-1)$ statistics, $S_1^2/S_k^2, \cdots, S_{k-1}^2/S_k^2$. Let $u_\alpha = S_\alpha^2/S_k^2$ and $v_\alpha = \sigma_\alpha^2/\sigma_k^2$ for $\alpha = 1, 2, \cdots, k-1$. Moreover, we set

$\overline{D}_{00}$: the decision that $v_1 = v_2 = \cdots = v_{k-1} = 1$

$\overline{D}_{ij}$: the decision that $v_1 = \cdots = v_{i-1} = v_{i+1} = \cdots = v_{j-1} = v_{j+1} = \cdots = v_{k-1} = 1$ and $v_i^2 = v_j^2 = \lambda^2$, $(i<j,\ i=1, \cdots, k-2,\ j \neq k)$

$\overline{D}_{ik}$: the decision that $v_1 = \cdots = v_{i-1} = v_{i+1} = \cdots = v_{k-1} = 1/\lambda^2$ and $v_i = 1$, $(i = 1, 2, \cdots, k-1)$

Then any allowable procedure for selecting one of the set $(D_{00}, D_{12}, \cdots, D_{k-1,k})$ can be transformed into a procedure for selecting one of the set $(\overline{D}_{00}, \overline{D}_{12}, \cdots, \overline{D}_{k-1,k})$.

According to the Truax's methods [4], it is necessary for obtaining the optimum solution to calculate the region consisting of all points, for which $\overline{D}_{\alpha\beta}$ is selected, in the $u_1, u_2, \cdots, u_{k-1}$ space where $p_{\alpha\beta} g_{\alpha\beta} = \max (p_{00} g_{00}, p_{12} g_{12}, \cdots, p_{k-1,k} g_{k-1,k})$ where $g_{\alpha\beta}$ is the joint probability density function of $u_1, u_2, \cdots, u_{k-1}$ when $\overline{D}_{\alpha\beta}$ is true and $p_{\alpha\beta}$ denotes the a priori probability that $\overline{D}_{\alpha\beta}$ is true. As we consider, under the restriction (ii), the situation that there is no basis, prior to the analysis of data, for determining which populations have been slipped to the right, we may put

$$p_{00} \equiv \left\{1 - \frac{k(k-1)}{2} p\right\},\quad p_{12} = p_{13} = \cdots = p_{k-1,k} \equiv p.$$

Then we can easily obtain the following results: the region for selecting $\overline{D}_{ij} (j \neq k)$ consists of all points in the $u_1, u_2, \cdots, u_{k-1}$ space which satisfy the relations :

$$u_i,\ u_j > 1,$$
$$u_i,\ u_j > \max (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_{j-1}, u_{j+1}, \cdots, u_{k-1}),$$
$$(u_i + u_j) / \{ \textstyle\sum_{\alpha=1}^{k-1} u_\alpha + 1 \} > L,$$

where $$L=\frac{\lambda^2}{\lambda^2-1}\left[1-\left\{\frac{p}{\left(1-\frac{k(k-1)}{2}p\right)(\lambda^2)^{(n-1)}}\right\}^{2/k(n-1)}\right],$$

and the region for selecting $\overline{D}_{ik}$ $(i=1, 2,\cdots, k-1)$ is

$$1>u_\alpha \ (\alpha=1,\cdots, i-1,\ i+1,\cdots, k-1),$$
$$u_i>\max(u_1,\cdots,u_{i-1},\ u_{i+1},\ \cdots,\ u_{k-1}),$$
$$(1+u_i)/\{\textstyle\sum_{\alpha=1}^{k-1}u_\alpha+1\}>L,$$

and finally the region for selecting $\overline{D}_{00}$ is the remainder in the u_1, $u_2,\cdots, u_{k-1}$ space. If we return to the initial notations by substituting $u_\alpha=S_\alpha^2/S_k^2$ and by making $\overline{D}_{ij}$ correspond to D_{ij}, we can obtain the optimum statistical procedure for selecting one of the $\frac{k(k-1)}{2}+1$ decisions, D_{00}, $D_{12},\cdots, D_{k-1,k}$, when two populations have been slipped to the right, that is :

if $(S_M^2+S_{M'}^2)/\sum_{i=1}^k S_i^2>L$, select $D_{MM'}$ and

if $(S_M^2+S_{M'}^2)/\sum_{i=1}^k S_i^2\leq L$, select D_{00},

where M, M' denote the populations yielding the two largest variance estimates. Here, because of the condition (i) we may replace L by L_α where L_α is the 100α percentage point of $(S_M^2+S_{M'}^2)/\sum_{i=1}^k S_i^{2'}$.

In case when the two populations have been slipped to the left, similar discussion can be made.

3. The distribution of $G(k, \nu)$. In this section, we shall derive the sampling distribution function of $G(k, \nu)$ in the null case, i. e. in the case when the variances of populations are all equal. As $S(k, \nu)=1-G(k, k-\nu)$, the distribution of $S(k, \nu)$ can be obtained from the distribution of $G(k, \nu)$ for $1\leq\nu\leq k-1$.

Let $x_1\leq x_2\leq\cdots\leq x_k$ be the ordered set of k random variables which are distributed independently as $\chi^2\sigma^2/n$ with n degrees of freedom each, where σ^2 is the common variance in the null case. Then, the statistic $G(k, \nu)$ is written as

$$G(k, \nu)=\frac{x_{k-\nu+1}+\cdots+x_k}{x_1+x_2+\cdots+x_k}. \tag{5}$$

The simultaneous density function of $x_1, x_2,\cdots, x_k$ is given by

$$f(x_1, x_2,\cdots, x_k)=\frac{k!}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}}\left(\frac{n}{2\sigma^2}\right)^{\frac{kn}{2}}(x_1\, x_2\cdots x_k)^{\frac{n}{2}-1} \tag{6}$$

$$\times \exp\left\{-\frac{n}{2\sigma^2}(x_1+x_2+\cdots+x_k)\right\}, \quad (x_1 \leqq x_2 \leqq \cdots \leqq x_k).$$

By making the transformation

$$(7) \qquad y_i = x_i/(x_1+x_2+\cdots+x_k), \qquad (i=1, 2, \cdots, k-1)$$
$$y_k = x_1+x_2+\cdots+x_k,$$

and integrating y_k over $0<y_k<\infty$, we obtain the joint density function of $y_1, y_2, \cdots, y_{k-1}$ with the form,

$$(8) \qquad f(y_1, y_2, \cdots, y_{k-1}) = k!\,\frac{\Gamma\left(\frac{kn}{2}\right)}{\left\{\Gamma\left(\frac{n}{2}\right)\right\}^k}(y_1 y_2 \cdots y_{k-1})^{\frac{n}{2}-1}$$
$$\times (1-y_1-y_2-\cdots-y_{k-1})^{\frac{n}{2}-1},$$

where y_i are restricted by the relations

$$(9) \qquad 0<y_1 \leqq y_2 \leqq \cdots \leqq y_{k-1} \leqq 1-y_1-y_2-\cdots-y_{k-1}.$$

Since $G(k, \nu)=1-y_1-y_2-\cdots-y_{k-\nu}$, we make the following transformation

$$(10) \qquad \begin{aligned}
y_1 &= (1-G)(1-u_{k-\nu})(1-u_{k-\nu-1})\cdots(1-u_3)(1-u_2),\\
y_2 &= (1-G)(1-u_{k-\nu})(1-u_{k-\nu-1})\cdots(1-u_3)u_2,\\
y_3 &= (1-G)(1-u_{k-\nu})(1-u_{k-\nu-1})\cdots(1-u_4)u_3,\\
&\vdots\\
y_{k-\nu-1} &= (1-G)(1-u_{k-\nu})u_{k-\nu-1},\\
y_{k-\nu} &= (1-G)u_{k-\nu},\\
y_{k-\nu+1} &= Gv_{k-\nu+1},\\
y_{k-\nu+2} &= G(1-v_{k-\nu+1})v_{k-\nu+2},\\
&\vdots\\
y_{k-2} &= G(1-v_{k-\nu+1})(1-v_{k-\nu+2})\cdots(1-v_{k-3})v_{k-2},\\
y_{k-1} &= G(1-v_{k-\nu+1})(1-v_{k-\nu+2})\cdots(1-v_{k-3})(1-v_{k-2})v_{k-1},
\end{aligned}$$

where $G=G(k, \nu)$. Then, since the Jacobian of this transformation is

$$(11) \qquad G^{\nu-1}(1-G)^{k-\nu-1}\prod_{\alpha=3}^{k-\nu}(1-u_\alpha)^{\alpha-2}\prod_{\beta=k-\nu+1}^{k-1}(1-v_\beta)^{k-1-\beta}$$

and $1-y_1-y_2-\cdots-y_{k-1}=G(1-v_{k-\nu+1})(1-v_{k-\nu+2})\cdots(1-v_{k-1})$, we can write the joint density function of $G, u_2, \cdots, u_{k-\nu}, v_{k-\nu+1}, \cdots, v_{k-1}$ in the following way:

$$(12) \qquad f(G, u_2, \cdots, u_{k-\nu}, v_{k-\nu+1}, \cdots, v_{k-1})$$
$$= k!\,\beta\left(G; \frac{\nu n}{2}, \frac{(k-\nu)n}{2}\right)\beta\left(u_2; \frac{n}{2}, \frac{n}{2}\right)\beta\left(u_3; \frac{n}{2}, n\right)$$

$$\times \cdots \times \beta\left(u_{k-\nu};\ \frac{n}{2},\ \frac{(k-\nu-1)n}{2}\right)\cdot\beta\left(v_{k-\nu+1};\ \frac{n}{2},\ \frac{(\nu-1)n}{2}\right)$$
$$\times\ \beta\left(v_{k-\nu+2};\ \frac{n}{2},\ \frac{(\nu-2)n}{2}\right)\cdots\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)$$

where $\beta(x;\ p,\ q)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1}$. Hence, in order to obtain the density function of $G(k,\ \nu)$, we need to integrate out $u_2,\cdots,u_{k-\nu},\ v_{k-\nu+1},\ \cdots,v_{k-1}$ from (12). The domain of integration is easily obtained from (9) and (10) as follows:

for $\nu=1$,
$$\frac{u_{k-1}}{1+u_{k-1}}<G<1,$$
$$\text{(13)}\qquad \frac{u_{i-1}}{1+u_{i-1}}<u_i<1, \qquad (i=k-1,\ k-2,\cdots,3)$$
$$\frac{1}{2}<u_2<1,$$

and for $2\leqq\nu\leqq k-1$,
$$\frac{v_{i-1}}{1-v_{i-1}}<v_i<\frac{1}{k-i+1}, \qquad (i=k-1,\cdots,k-\nu+2)$$
$$\frac{1-G}{G}u_{k-\nu}<u_{k-\nu+1}<\frac{1}{\nu},$$
$$\text{(14)}\qquad \frac{u_{j-1}}{1+u_{j-1}}<u_j<h_j(G), \quad (j=k-\nu,\ k-\nu-1,\cdots,3)$$
$$\frac{1}{2}<u_2<h_2(G),$$
$$\frac{\nu}{k}<G<1,$$

where $h_j(G)$ is a function of G and j such that it takes the value 1 for $1>G>\nu/(k-j+1)$ and equals to $G/[\nu-(k-j)G]$ for $\nu/k<G<\nu/(k-j+1)$.

If we define the function $W_r(x)$ such that

$$\text{(15)}\qquad W_0(x)=1, \quad W_r(x)=(r+1)\int_x^{\frac{1}{r+1}}\beta\left(t:\ \frac{n}{2},\ \frac{rn}{2}\right)W_{r-1}\left(\frac{t}{1-t}\right)dt.$$

we can write the simultaneous density function of $G,\ u_2,\cdots,u_{k-\nu}$ after integrating out $v_{k-1},\ v_{k-2},\cdots,v_{k-\nu}$, in the form

$$\text{(16)}\qquad f(G,\ u_2,\cdots,u_{k-\nu})$$
$$=\frac{k!}{\nu!}\beta\left(G;\ \frac{\nu n}{2},\ \frac{(k-\nu)n}{2}\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)$$

$$\times \cdots\cdots \times \beta\left(u_{k-\nu};\ \frac{n}{2},\ \frac{(k-\nu-1)n}{2}\right) W_{\nu-1}\left(\frac{1-G}{G} u_{k-\nu}\right).$$

4. Special cases (I). In this section, we shall discuss special cases of the sampling distribution functions when $\nu=1$ and $\nu=k-1$. First we shall consider the case where $\nu=1$, i.e. Cochran's case. In this case, in order to obtain the distribution function of $G(k, 1)$, we must change the order of the integration in the inverse way.

$$k! \int_{\frac{1}{2}}^{1} \beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right) du_2 \int_{\frac{u_2}{1+u_2}}^{1} \beta\left(u_3;\ \frac{n}{2},\ n\right) du_3 \cdots \tag{17}$$
$$\times \int_{\frac{u_{k-2}}{1+u_{k-2}}}^{1} \beta\left(u_{k-1};\ \frac{n}{2},\ \frac{(k-2)n}{2}\right) du_{k-1} \int_{\frac{u_{k-1}}{1+u_{k-1}}}^{1} \beta\left(G;\ \frac{n}{2},\ \frac{(k-1)n}{2}\right) dG = 1$$

Changing the order of the integration, for $k=3$,

$$1 = 3!\left[\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} + \int_{\frac{1}{3}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{G}{1-G}}\right] \beta\left(G;\ \frac{n}{2},\ n\right) \beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right) du_2 dG. \tag{*}$$

Hence, we have

$$P(G>g) = 3I_{1-g}\left(n,\ \frac{n}{2}\right) \tag{18}$$

for $1>g\geqq\frac{1}{2}$ and

$$P(G>g) = 3I_{1-g}\left(n,\ \frac{n}{2}\right) \tag{19}$$
$$-3\cdot 2\int_{g}^{\frac{1}{2}}\int_{\frac{G}{1-G}}^{1} \beta\left(G;\ \frac{n}{2},\ n\right)\beta\left(u_2;\ \frac{h}{2},\ \frac{n}{2}\right) du_2 dG$$

for $\frac{1}{2}\geqq g\geqq\frac{1}{3}$, where $I_x(p, q)$ is the K. Pearson's incomplete beta function ratio. When $k=4$.

$$1 = 4!\left[\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} + \int_{\frac{1}{3}}^{1}\int_{\frac{1}{3}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{u_3}{1-u_3}} + \int_{\frac{1}{3}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{G}{1-G}}\int_{\frac{1}{2}}^{1} + \int_{\frac{1}{4}}^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{G}{1-G}}\int_{\frac{1}{2}}^{\frac{u_3}{1-u_3}}\right] \times$$
$$\times \beta\left(G;\ \frac{n}{2},\ \frac{3n}{2}\right) \beta\left(u_3;\ \frac{n}{2},\ n\right) \beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right) du_2 du_3 dG.$$

Then, noticing the relation (*), we obtain

$$P(G>g) = 4I_{1-g}\left(\frac{3n}{2},\ \frac{n}{2}\right) \tag{20}$$

for $1>g\geqq\frac{1}{2}$ and

$$P(G>g)=4I_{1-g}\left(\frac{3n}{2},\ \frac{n}{2}\right)$$

(21)
$$-4\cdot3\int_{g}^{\frac{1}{2}}\int_{\frac{G}{1-G}}^{1}\beta\left(G;\ \frac{n}{2},\ \frac{3n}{2}\right)\beta\left(u_3;\ \frac{n}{2},\ n\right)du_3dG,$$

for $\frac{1}{2}\geqq g\geqq\frac{1}{3}$ and

$$P(G>g)=4I_{1-g}\left(\frac{3n}{2},\ \frac{n}{2}\right)$$

(22)
$$-4\cdot3\int_{g}^{\frac{1}{2}}\int_{\frac{G}{1-G}}^{1}\beta\left(G;\ \frac{n}{2},\ \frac{3n}{2}\right)\beta\left(u_3;\ \frac{n}{2},\ n\right)du_3dG$$

$$+4\cdot3\cdot2\int_{g}^{\frac{1}{3}}\int_{\frac{G}{1-G}}^{\frac{1}{2}}\int_{\frac{u_3}{1-u_3}}^{1}\beta\left(G;\ \frac{n}{2},\frac{3n}{2}\right)\beta\left(u_3;\ \frac{n}{2},n\right)\beta\left(u_2;\frac{n}{2},\frac{n}{2}\right)du_2du_3dG.$$

for $\frac{1}{3}\geqq g\geqq\frac{1}{4}$.

In the analogous way, we can generally write the distribution function of $G(k,\ 1)$ for k in the following form:

(23) $P(G>g)=kP_1(g),$ for $1>g\geqq\frac{1}{2}$,

(24) $P(G>g)=kP_1(g)-k(k-1)P_2(g),$ for $\frac{1}{2}\geqq g\geqq\frac{1}{3}$,

....

(25) $P(G>g)=kP_1(g)-k(k-1)P_2(g)+\cdots\cdot+(-1)^{j+1}k(k-1)\cdots\cdot$
$\cdots\cdot(k-j+1)P_j(g)$

for $1/j\geqq g\geqq1/(j+1)$, and

(26) $P(G>g)=kP_1(g)-k(k-1)P_2(g)+\cdots\cdot+(-1)^{k}k(k-1)\cdots\cdot3\cdot2P_{k-1}(g),$

for $1/(k-1)\geqq g\geqq\frac{1}{k}$, where

(27) $$P_1(g)=\int_{g}^{1}\beta\left(G;\ \frac{n}{2},\ \frac{(k-1)n}{2}\right)dG=I_{1-g}\left(\frac{(k-1)n}{2},\ \frac{n}{2}\right),$$

and

(28) $$P_j(g)=\int_{g}^{\frac{1}{j}}\int_{\frac{G}{1-G}}^{\frac{1}{j-1}}\int_{l_{k-1}}^{\frac{3}{j-2}}\cdots\cdot\int_{l_{k-j+3}}^{\frac{1}{2}}\int_{l_{k-j+2}}^{1}\beta\left(G;\ \frac{n}{2},\ \frac{(k-1)n}{2}\right)$$

$$\times\beta\left(u_{k-1};\ \frac{n}{2},\ \frac{(k-2)n}{2}\right)\cdots\cdot\beta\left(u_{k-j+2};\ \frac{n}{2},\ \frac{(k-j+1)n}{2}\right)$$
$$\times\beta\left(u_{k-j+1};\ \frac{n}{2},\ \frac{(k-j)n}{2}\right)du_{k-j+1}\cdots\cdot du_{k-1}dG\,,$$

where we put $l_\alpha=u_\alpha/(1-u_\alpha)$.

Next we shall consider the case when $\nu=k-1$. The sampling distribution function of $G(k,\ k-1)=1-S(k,\ 1)$ can be obtained from (16) in the following simple form:

$$\begin{aligned}P(G>g)&=P(S<1-g)\\&=k\int_g^1\beta\left(G;\ \frac{(k-1)n}{2},\ \frac{n}{2}\right)W_{k-2}\left(\frac{1-G}{G}\right)dG\\&=1-W_{k-1}(1-g)\end{aligned}\tag{29}$$

By subdividing the integration, we can write the distribution function of $G(k,\ k-1)$ in the following expanded form, i.e.

$$\begin{aligned}P(G>g)=&kP_1^*(g)-k(k-1)P_2^*(g)+\cdots\cdot\\&+(-1)^k k(k-1)\cdots\cdot3\cdot2P_{k-1}^*(g),\end{aligned}\tag{30}$$

where

$$P_1^*(g)=\int_0^{1-g}\beta\left(S;\ \frac{n}{2},\ \frac{(k-1)n}{2}\right)dS=I_{1-g}\left(\frac{n}{2},\ \frac{(k-1)n}{2}\right),\tag{31}$$

$$\begin{aligned}P_j^*(g)=&\int_0^{1-g}\int_0^{\frac{S}{1-S}}\int_0^{l_2'}\cdots\cdot\int_0^{l'_{j-1}}\beta\left(S;\ \frac{n}{2},\ \frac{(k-1)n}{2}\right)\\&\times\beta\left(v_2;\ \frac{n}{2},\ \frac{(k-2)n}{2}\right)\cdots\cdot\beta\left(v_j;\ \frac{n}{2},\ \frac{(k-j)n}{2}\right)dv_j\cdots\cdot dv_2dS\end{aligned}\tag{32}$$

for $j=2,\ 3,\cdots\cdot,\ k-1$, where $l'_\alpha=v_\alpha/(1-v_\alpha)$.

It should be noticed that the expanded forms (25) and (32) have been made in such a way that the true value of $P(G>g)$ exists between the values obtained from the first m terms and $m+1$ terms, where $1\leq m\leq k-1$, and as will be seen in Section 6, we shall make use of this fact in order to obtain the approximate 5% and 1% points of $G(k,\ 1)$ and $S(k,\ 1)$ for each k and n.

5. Special cases (II). As in the preceding section, we shall consider the expanded forms of the distribution functions of $G(k,\ 2)$ and $G(k,\ k-2)$ or $S(k,\ 2)$ to evaluate the 5% and 1% points. We can easily write the distribution function of $G(k,\ k-2)=1-S(k,\ 2)$ from (16) as follows:

$$
(33)\qquad 1=k(k-2)\left[\int_{\frac{k-2}{k-1}}^{1}\int_{\frac{1}{2}}^{1}+\int_{\frac{k-2}{k}}^{\frac{k-2}{k-1}}\int_{\frac{1}{2}}^{\frac{G}{(k-2)(1-G)}}\right]\beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\times\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)W_{k-3}\left(\frac{1-G}{G}u_2\right)\cdot du_2dG
$$

where $G\equiv G(k,\ k-2)$, Hence we obtain

$$
(34)\qquad P(G>g)=k(k-1)\int_{g}^{1}\int_{\frac{1}{2}}^{1}\beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)\times W_{k-3}\left(\frac{1-G}{G}u_2\right)du_2dG
$$

for $1>g\geqq(k-2)/(k-1)$, and

$$
(35)\qquad P(G>g)=k(k-1)\left[\int_{g}^{1}\int_{\frac{1}{2}}^{1}-\int_{g}^{\frac{k-2}{k-1}}\int_{\frac{G}{(k-2)(1-G)}}^{1}\right]\beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\times\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)W_{k-3}\left(\frac{1-G}{G}u_2\right)\cdot du_2dG
$$

for $(k-2)/(k-1)\geqq g\geqq(k-2)/k$. If we write the expanded form of $W_{k-3}\left(\frac{1-G}{G}u_2\right)$ in the analogous way as (30), we have

$$
W_{k-3}\left(\frac{1-G}{G}u_2\right)=1-(k-2)P_2'(G,\ u_2)+(k-2)(k-3)P_3'(G,\ u_2)-\cdots\cdot+(-1)^{k-1}(k-2)(k-3)\cdots\cdot3\cdot2P_{k-2}'(G,\ u_2)
$$

where

$$
P_2'(G,\ u_2)=\int_0^{\frac{1-G}{G}u_2}\beta\left(v_3;\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)dv_3\ ,
$$

and

$$
P_j'(G,\ u_2)=\int_0^{\frac{1-G}{G}u_2}\int_0^{l_3'}\int_0^{l_4'}\cdots\cdot\int_0^{l_j'}\beta\left(v_3;\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)\times\beta\left(v_4;\ \frac{n}{2},\ \frac{(k-4)n}{2}\right)\cdots\cdot\beta\left(v_j;\ \frac{n}{2},\ \frac{(k-j)n}{2}\right)\times\beta\left(v_{j+1};\ \frac{n}{2},\ \frac{(k-j\ 1)n}{2}\right)dv_{j+1}\cdots\cdot dv_4dv_3\ ,
$$

for $j=2,\ 3,\cdots\cdot,\ k-2$, where $l_\alpha'=v_\alpha/(1-v_\alpha)$. Substituting this expanded form into (34) and (35), we have

$$
(36)\qquad P(G>g)=k(k-1)[Q_1^{(1)}(g)-Q_1^{(2)}(g)]-k(k-1)(k-2)[Q_2^{(1)}(g)-Q_2^{(2)}(g)]+\cdots\cdot+(-1)^{k-1}k(k-1)\cdots\cdot3\cdot2[Q_{k-2}^{(1)}(g)-Q_{k-2}^{(2)}(g)]\ ,
$$

where

$$Q_1^{(1)}(g)=\int_g^1\int_{\frac{1}{2}}^1 \beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)du_2dG$$

$$=\frac{1}{2}I_{1-g}\left(n,\ \frac{(k-2)n}{2}\right), \tag{37}$$

$$Q_1^{(2)}(g)=\int_g^{\frac{k-2}{k-1}}\int_{\frac{G}{(k-2)(1-G)}}^1 \beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)du_2dG$$

and

$$Q_j^{(1)}(g)=\int_g^1\int_{\frac{1}{2}}^1 \beta\left(G;\ \frac{(k-2)n}{2},\ n\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)P_j'(G,\ u_2)du_2dG,$$

$$Q_j^{(2)}(g)=\int_g^{\frac{k-2}{k-1}}\int_{\frac{G}{(k-2)(1-G)}}^1 \beta\left(G;\ \frac{(k-2)n}{2},n\right)\beta\left(u_2,\ \frac{n}{2},\ \frac{n}{2}\right)P_j'(G,\ u_2)du_2dG \tag{38}$$

for $j=2, 3,\cdots, k-2$ and we put $Q_j^{(2)}(g)=0$ $(j=1, 2,\cdots, k-2)$ whenever $1>g\geqq\frac{k-2}{k-1}$.

When $\nu=2$, the distribution functions of $G(k, 2)$ are given, from (12) and (14) as follows:

$$P(G>g)=R_1(g), \qquad \text{for } 1>g\geqq 2/3 \tag{39}$$

$$P(G>g)=R_1\left(\frac{2}{3}\right)+R_2(g), \qquad \text{for } 2/3>g\geqq 2/4 \tag{40}$$

and generally

$$P(G>g)=R_1\left(\frac{2}{3}\right)+R_2\left(\frac{2}{4}\right)+\cdots+R_{j-3}\left(\frac{2}{j-1}\right)+R_{j-2}(g) \tag{41}$$

for $2/(j-1)\geqq g\geqq 2/j$, where j is a positive integer such that $k\geqq j\geqq 4$ and

$$R_1(g)=k!\int_g^1\int_{\frac{1}{2}}^1\int_{w_2}^1\cdots\int_{w_{k-3}}^1\int_{\frac{1-G}{G}u_{k-2}}^1 H(G,\ u_2,\cdots u_{k-2},\ v_{k-1})dv_{k-1}\cdots du_2dG,$$

$$R_\alpha(g)=k!\int_g^{\frac{2}{\alpha+2}}\int_{\frac{1}{2}}^1\int_{w_2}^1\cdots\int_{w_{k-\alpha-2}}^1\int_{w_{k-\alpha-1}}^{\frac{G}{2-\alpha G}}\int_{w_{k-\alpha}}^{\frac{G}{2-(\alpha-1)G}}\cdots$$

$$\int_{w_{k-3}}^{\frac{G}{2-2G}}\int_{\frac{1-G}{G}u_{k-2}}^{\frac{1}{2}} H(G,\ u_2,\cdots u_{k-2},\ v_{k-1})\cdot dv_{k-1}\cdots du_2dG\ ,$$

$$(\alpha=2,\ 3,\cdots,\ k-2)$$

where we put $w_i=u_i/(1+u_i)$ and

$$H(G,\ u_2,\cdots,\ u_{k-1})=\beta\left(G;\ n,\ \frac{(k-2)n}{2}\right)\beta\left(u_2;\ \frac{n}{2},\ \frac{n}{2}\right)\cdots$$

$$\times\beta\left(u_{k-2};\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)\cdot\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)$$

Each $R_j(g)$, for $j=1, 2, \cdots, k-2$, can be expanded, though cumbersome, in the analogous way as the previous cases by dividing the integration. That is,

(42) $$R_1(g)=k(k-1)R_{11}(g)-k(k-1)(k-2)R_{12}(g)+\cdots\cdot+(-1)^{k-1}k(k-1)\cdots\cdot 3\cdot 2R_{1(k-2)}(g),$$

where

$$R_{11}(g)=\frac{1}{2}I_{1-g}\left(\frac{(k-2)n}{2},\ n\right),$$

$$R_{12}(g)=\int_g^1\int_{\frac{1}{k-2}}^1\int_0^{\frac{1-G}{G}u_{k-2}}\beta\left(G;\ n,\ \frac{(k-2)n}{2}\right)\beta\left(u_{k-2};\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)\times\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)dv_{k-1}du_{k-2}dG$$

and

$$R_{1\alpha}(g)=\int_g^1\int_{\frac{1}{k-\alpha}}^1\int_{\frac{1}{k-\alpha+1}}^{w_{k-\alpha}}\cdots\cdot\int_{\frac{1}{k-2}}^{w_{k-3}}\int_0^{\frac{1-G}{G}u_{k-2}}\beta\left(G;\ n,\ \frac{(k-2)n}{2}\right)\times\beta\left(u_{k-\alpha};\ \frac{n}{2},\ \frac{(k-\alpha-1)n}{2}\right)\cdots\cdot\beta\left(u_{k-2};\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)\times\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)dv_{k-1}du_{k-2}\cdots\cdot du_{k-\alpha}dG,$$

for $\alpha=3,\cdots\cdot, k-2$, and

(43) $$R_2(g)=k(k-1)(k-2)R_{22}(g)-k(k-1)(k-2)(k-3)R_{23}(g)+\cdots\cdot+(-1)^{k-2}k(k-1)\cdots 3\cdot 2R_{2(k-2)}(g),$$

where

$$R_{22}(g)=\int_g^{\frac{2}{3}}\int_{\frac{1}{k-2}}^{\frac{G}{2-2G}}\int_{\frac{1-G}{G}u_{k-2}}^{\frac{1}{2}}\beta\left(G;\ n,\ \frac{(k-2)n}{2}\right)\beta\left(u_{k-2};\ \frac{n}{2},\ \frac{(k-3)n}{2}\right)\times\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)dv_{k-1}du_{k-2}dG,$$

and

$$R_{2\alpha}(g)=\int_g^{\frac{2}{3}}\int_{\frac{1}{k-\alpha}}^1\int_{\frac{1}{k-\alpha+1}}^{w_{k-\alpha}}\cdots\cdot\int_{\frac{1}{k-2}}^{w_{k-3}}\int_{\frac{1-G}{G}u_{k-2}}^{\frac{1}{2}}\beta\left(G;\ n, \frac{(k-2)n}{2}\right)\times\beta\left(u_{k-2};\ \frac{n}{2},\ \frac{(k-\alpha-1)n}{2}\right)\cdots\cdot\beta\left(u_{k-2};\ \frac{n}{2};\ \frac{(k-3)n}{2}\right)\times\beta\left(v_{k-1};\ \frac{n}{2},\ \frac{n}{2}\right)dv_{k-1}du_{k-2}\cdots\cdot du_{k-\alpha}dG$$

$(\alpha=3,\cdots\cdot,\ k-2)$ and so on.

Since the exact expressions of the distribution functions of $G(k, \nu)$, though we can write them for each k and ν as is seen above, are too cumbersome and too labourious to compute the percentage points of the significance, we shall study, in the following section, the approximate evaluation of the numerical distribution functions about the neighbourhood of the 5% and 1% points of significance.

6. Approximate evaluation of the percentage points of $G(k, 1)$, $G(k, 2)$, $S(k, 1)$ **and** $S(k, 2)$**.** In order to obtain the upper 5% and 1% points of significance of $G(k, 1)$ and $G(k, 2)$ and the lower 5% and 1% points of significance of $S(k, 1)$ and $S(k, 2)$, we shall examine numerically the degrees of approximation of evaluation for 5% points which are obtained using the first term, first two terms, etc. of the expanded forms such as in the preceding sections.

First, we shall examine the cases of $G(k, 1)$ and $S(k, 1)$. In these cases, the first term and the first two terms of the distribution functions are

$$P(G>g)\approx kI_{1-g}\left(\frac{(k-1)n}{2}, \frac{n}{2}\right), \tag{44, a}$$

$$P(G>g)\approx kI_{1-g}\left(\frac{(k-1)n}{2}, \frac{n}{2}\right)-k(k-1)P_2(g) \tag{44, b}$$

for $G(k, 1)$ and

$$P(S<s)\approx kI_s\left(\frac{n}{2}, \frac{(k-1)n}{2}\right), \tag{45, a}$$

$$P(S<s)\approx kI_s\left(\frac{n}{2}, \frac{(k-1)n}{2}\right)-k(k-1)P_2^*(1-s) \tag{45, b}$$

for $S(k, 1)$, where (44, a) is exact whenever $g\geqq\frac{1}{2}$ and (44, b) is exact whenever $g\geqq 1/3$.

Tables IA and IB show the uper 5% points of $G(k, 1)$ and the lower 5% points of $S(k, 1)$, i.e. the values $g(0.05)$ such that $0.05=P\{G(k, 1)>g(0.05)\}$ and the values $s(0.05)$ such that $0.05=P\{S(k, 1)<s(0.05)\}$ using the formulas (44) and (45), respectively.

The values of $g(0.05)$ and $s(0.05)$ in Tables IA and IB were obtained with the aid of the polynomials in g and s, to which (44) and (45) can be reduced, and of Tables of Incomplete Beta Function [5], for appropriate lower values of n degrees of freedom. For higher values of n,

Table 1 A. The upper 5% points of $G(k, 1)-g(0.05)-$ using (44, a) and (45, b)

k \ n		2	6	10	20	30	50	100
4	(44, a)	0.76792	0.55980	0.48838	0.41594	0.38416	0.35272	0.34775
	(44, b)	* *	* *	0.48838	0.41594	0.38415	0.35271	0.34774
10	(44, a)	0.44495	0.28228	0.23534	0.19110	0.17248	0.15460	0.13747
	(44, b)	0.44495	0.28224	0.23528	0.19103	0.17241	0.15454	0.13740
20	(44, a)	0.27046	0.16023	0.13044	0.10326	0.09204	0.08141	0.07138
	(44, b)	0.27041	0.16017	0.13038	0.10321	0.09198	0.08136	0.07133

Table 1 B. The lower 5% points of $S(k, 1)-s(0.05)-$ using (45, a) and (45, b)

k \ n		2	6	10	20	30	50	100
4	(45, a)	0.00418	0.04647	0.07685	0.11713	0.13800	0.16060	0.18507
	(45, b)	0.00424	0.04656	0.07693	0.11720	0.13806	0.16065	0.18512
10	(45, a)	0.00056	0.01200	0.02261	0.03836	0.04729	0.05732	0.06852
	(45, b)	0.00059	0.01205	0.02267	0.03842	0.04736	0.05738	0.06858
20	(45, a)	0.00013	0.00453	0.00901	0.01699	0.02154	0.02677	0.03273
	(45, b)	0.00016	0.00456	0.00904	0.01703	0.02158	0.02682	0.03278

they were evaluated from the values of the F-distribution which were determined by means of E. Paulson's procedure [6]. $P_2(g)$ and $P_2^*(1-s)$ were evaluated by the relations

$$(46)\quad P_2(g)\approx\lambda\int_g^{\frac{1}{2}}\beta\Big(G;\ \frac{n}{2},\ \frac{(k-1)n}{2}\Big)dG\int_{\frac{g}{1-g}}^{1}\beta\Big(u;\ \frac{n}{2},\ \frac{(k-2)n}{2}\Big)du,$$

$$=\lambda\Big\{I_{1-g}\Big(\frac{(k-1)n}{2},\ \frac{n}{2}\Big)-I_{\frac{1}{2}}\Big(\frac{(k-1)n}{2},\ \frac{n}{2}\Big)\Big\}\cdot$$

$$\times I_{\frac{1-2g}{1-g}}\Big(\frac{(k-2)n}{2},\ \frac{n}{2}\Big),$$

and

$$(47)\quad P_2^*(1-s)\approx\lambda^*\Big\{1-I_{1-s}\Big(\frac{(k-1)n}{2},\ \frac{n}{2}\Big)\Big\}\Big\{1-I_{\frac{1-2s}{1-s}}\Big(\frac{(k-2)n}{2},\ \frac{n}{2}\Big)\Big\}$$

where λ and λ^* are about 1/2, but are chosen in such a way that the right-hand sides of (46) and (47) are not smaller than the left-hand sides with the aid of the exact values obtained from cases of $n=1, 2, \cdots, 10$.

Since the exact 5% points, $g(0.05)$ and $s(0.05)$, of signfıcance of $G(k, 1)$ and $S(k, 1)$ lie, as is noticed in Section 4, between the two values obtained from two expressions in (44) and (45), respectively, it will be recognized that the 5% and 1% levels of $G(k, 1)$ and $S(k, 1)$ can be evaluated to a good approximation using only the first term of the expanded forms of their distribution functions, i.e. (44, a) and (45, a) and the values thus obtained are accurate up to four decimal places.

Now we shall examine the degree of approximation in the analogous way, when the 5% points of $S(k, 2)$ are evaluated. The expression to be examined is

$$(48)\qquad \begin{aligned} P(S<s) &= k(k-1)\{Q_1^{(1)}(1-s)-Q_1^{(2)}(1-s)\}-k(k-1)(k-2)\{Q_2^{(1)}(1-s) \\ &\quad -Q_2^{(2)}(1-s)\}+k(k-1)(k-2)(k-3)\{Q_3^{(1)}(1-s)-Q_3^{(2)}(1-s)\} \\ &= (1)-(2)+(3) \end{aligned}$$

where $Q_j^{(2)}(1-s)=0$, $(j=1, 2, 3)$ whenever $s\leqq \dfrac{1}{k-1}$. Table 2 A shows the lower 5% points of $S(k, 2)$ using the formula (48) up to (1), (2), and (3).

Table 2 A. The lower 5% points of $S(k, 2)-s(0.05)-$using (52) up to (1), (2) and (3)

k \ n		2	6	10	16	20
5	(1)	0.02945	0.12668	0.17590	0.21655	0.23389
	(1) − (2)	0.03091	0.12855	0.17739	0.21782	0.23503
	(1) − (2) + (3)	0.03093	0.12853	0.17737	0.21780	0.23502
10	(1)	0.00563	0.04186	0.06543	0.08679	0.09607
	(1) − (2)	0.00619	0.04341	0.06703	0.08831	0.09754
	(1) − (2) + (3)	0.00614	0.04328	0.06690	0.08818	0.09742
20	(1)	0.00125	0.01504	0.02595	0.03625	0.04126
	(1) − (2)	0.00141	0.01581	0.02688	0.03739	0.04244
	(1) − (2) + (3)	0.00139	0.01568	0.02676	0.03726	0.04230

Here (2) and (3) have been calculated from the polynomials in s to which they are reduced. From these results, it can be seen that, in order to obtain the 5% and 1% points of $S(k, 2)$ which are accurate up to four decimal places, we need to use the first two terms of the expanded form, i. e. (1)−(2).

When we want to evaluate the 5% and 1% points of $G(k, 2)$, the labour of calculation is much increased. For example, when $k=5$ and n=10, in order to obtain the 5% point of G(5, 2), we must calculate $R_{11}\left(\frac{2}{3}\right)$, $R_{12}\left(\frac{2}{3}\right)$, $R_{13}\left(\frac{2}{3}\right)$ and $R_{22}(g)$. Thus, if calculation must be performed by using the expressions (39)~(41), it is very labourious to obtain the desired percentage points of $G(k, 2)$ for high values of k and n. The author has not been able to obtain the more appropriate expanded form of the distribution function of G(k, 2) than the expressions (39)~(41).

Tables of the upper 5% and 1% points, $g(0.05)$ and $g(0.01)$, of $G(k, 1)$ have been given in [2] for several values of k and n.

I shall give the tables of other cases in near future.

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