On the Poisson-Gamma Distribution Problem*

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Let

$$X_1, X_2, \cdots\cdots \tag{1}$$

be a sequence of non-negative variables independently distributed according to the same distribution function $F(x)$. Let x be any positive number, and N_x be a random variable defined as follows:

$$N_x = \begin{cases} 0 & \text{if} \quad X_1 > x \\ n & \text{if} \quad X_1+X_2+\cdots\cdots+X_n \leqq x \quad \text{and} \quad X_1+X_2+\cdots\cdots+X_{n+1} > x. \end{cases} \tag{2}$$

If $F(x)$ is the Gamma distribution function

$$F(x) = \begin{cases} 0 \quad \text{for} \quad x<0 \\ \displaystyle\int_0^x \frac{t^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{t}{\beta}} dt \quad \text{for} \quad x \geqq 0, \end{cases} \tag{3}$$

then N_x is, as is known, distributed according to the Generalized Poisson Law.

$$P_r\{N_x \leqq n\} = e^{-f(x)}\Big[1+f(x)+\frac{1}{2!}f^2(x)+\frac{1}{3!}f^3(x)+\cdots\cdots + \frac{1}{[(n+1)\alpha-1]!}f^{[(n+1)\alpha-1]}(x)\Big], \tag{4}$$

when α is a positive integer with $f(x)=\dfrac{x}{\beta}$.

Mr. Seiji Nabeya [1] has proven the converse of this fact when $\alpha=1$. In this paper we show that, when $F(x)$ is continuous, a somewhat simpler proof of the Nabeya result may be given which extends his theorem to the case when $\alpha \leqq 2$.

Theorem. Let (1) be a sequence of non-negative independent random variables with the same continuous distribution function $F(x)$, and let N_x be defined as (2). If N_x is distributed according to the Generalized Poisson Law (4), with fixed $\alpha \leqq 2$, for every positive x, then $F(x)$ is of the Gamma type (3).

Proof: Denoting the distribution function of $X_1+X_2+\cdots\cdots+X_n$ by $F_n(x)$ we have that

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$$F_n(x)=\begin{cases}0 \quad \text{for} \quad x<0,\\ 1-e^{-f(x)}\left[1+f(x)+\dfrac{f^2(x)}{2!}+\cdots\cdots+\dfrac{f^{n\alpha-1}}{(n\alpha-1)!}\right] \quad \text{for} \quad x\geqq 0.\end{cases} \tag{5}$$

Since,
$$F=1-e^{-g}\left[1+g+\cdots\cdots+\frac{g^{\alpha-1}}{(\alpha-1)!}\right]$$

is an increasing continuous function of g, the inverse function $g(F)$ is an increasing continuous function of F. Also $F=F_1(x)$ is a non-decreasing continuous function of x, and therefore, $g(F(x))=f(x)$ is a non-decreasing continuous function of x. Furthermore, $f(x)=f(y)$, for $x<y$, only when

$$P_r\{x<X_1+X_2+\cdots\cdots+X_n\leqq y\}=0.$$

Hence,

$$P_r\{f(x)<f\}=H_n(f)=\begin{cases}0 \quad \text{for} \quad x<0,\\ 1-e^{-f}\left[1+f+\cdots\cdots+\dfrac{f^{n\alpha-1}}{(n\alpha-1)!}\right] \quad \text{for} \quad x>0.\end{cases}$$

If $x[f]$ denotes the inverse function of f, then we have for $t>0$

$$\int_0^\infty \frac{f^{n\alpha-1}}{(n\alpha-1)!}e^{-tx[f]-f}df=\phi^n(t).$$

For fixed t, we regard

$$\frac{h^{\alpha-1}e^{-tx[h]-h}}{(\alpha-1)!\,\phi(t)}\,dh$$

as a probability density of the random variable h.

Hence we have that

$$E\{h^{\alpha(n-1)}\}=\frac{(n\alpha-1)!}{(\alpha-1)!}\phi^{n-1}(t),$$

and
$$E\left\{\left(\frac{h^\alpha}{\phi(t)}\right)^{n-1}\right\}=\frac{(n\alpha-1)!}{(\alpha-1)!}.$$

It is easy to see that if y is a random variable with probability density

$$\frac{e^{-y^{\frac{1}{\alpha}}}}{\alpha!}\,dy,$$

then its moments coincide with the moments of $\dfrac{h^\alpha}{\phi(t)}$.

For $\alpha\leqq 2$, it is well known [2] that the moment problem is determined. Hence we have that,

$$y^{\frac{1}{\alpha}}=tx[(y\phi(t))^{\frac{1}{\alpha}}]+(y\phi(t))^{\frac{1}{\alpha}},$$

$$x[(y\phi(t))^{\frac{1}{\alpha}}]=(y\phi(t))^{\frac{1}{\alpha}}\left[\frac{\phi(t)^{-\frac{1}{\alpha}}-1}{t}\right],$$

$$x[f]=f\left[\frac{\phi(t)^{-\frac{1}{\alpha}}-1}{t}\right].$$

This proves that

$$f(x)=\frac{t\phi(t)^{\frac{1}{\alpha}}}{1-\phi(t)^{\frac{1}{\alpha}}}x=\frac{x}{\beta}. \tag{6}$$

Substituting (6) in (5), we have the desired result.

REFERENCES

[1] Seiji NABEYA: "On a Relation between Exponential Law and Poisson's Law", *Annals of the Institute of Statistical Mathematics*, Vol. II, No. 1, Aug. 1950.

[2] J. A. SHOHAT and J. D. TAMARKIN: "The Problem of Moments", *Mathematical Surveys*, No. 1, American Mthematical Society.

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