On the Convergence of Classes of Distributions

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Introduction. Let $F_1(x)$, $F_2(x)$ be two one-demensional d. f. 's (distribution functions). If there are a positive number a and a real number b such that

$$F_2(x) = F_1(ax + b), \qquad (-\infty < x < \infty),$$

then, $F_1(x)$ and $F_2(x)$ are said to belong to the same class. Let K_0, K_1, K_2 .... be a sequence of classes of d. f. 's. If we can choose d. f. 's F_0, F_1, F_2,......from the classes K_0, K_1, K_2,......respectively, such that $\{ F_n(x) ; n = 1, 2, ... \}$ converges to $F_0(x)$ in every continuity point of $F_0(x)$, it is said that the sequence of classes of K_1, K_2, ... converges to the class K_0. A. I. Khinchine proved that if a given sequence of classes K_1, K_2,........ converges to a proper class, the limiting proper class is unique [2]. The main purpose of this note is to give a new simple proof to this fact by using the inverse functions of the d. f. 's. The principle of our proof seems very natural. The second is to give some expositions on the inverse functions of the d. f. 's.

§ 1. The inverse functions of the d. f. 's. Let $F(x)$ be a d. f., that is

(i) $F(x)$ is a non-decreasing function defined in the whole interval $-\infty < x < \infty$,

(ii) $\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to\infty} F(x) = 1$

(iii) $F(x)$ is continuous to the left: $F(x) = F(x-0)$, $-\infty < x < \infty$.

Define $f(y)$ as follows,

$$f(0) = -\infty,$$

$$f(y) = \min\{x\,;\, F(x+0) \geqq y\}, \qquad (\text{if } 0 < y < 1),$$

$$f(1) = \sup\{x\,;\, F(x) < 1\}.$$

If there exist finite x such that $F(x) = 1$, we have

$$f(1) = \min\{x\,;\, F(x+0) = 1\} = \inf\{x\,;\, F(x) = 1\}.$$

It is seen that

(i) $f(y)$ is non-decreasing,

(ii) $f(y)$ is continuous to the left in the open interval $0 < y < 1$; that is, $f(y)$ is finite and $f(y) = f(y-0)$ in $0 < y < 1$,

(iii) $f(1) = f(1-0)$,

Proof. (i) : evident. (ii) : It is sufficient to prove that $f(y) \leqq f(y-0)$.

For any positive number ε such that $0 < \varepsilon < y$, we have, by the definition of $f(y-\varepsilon)$,

$$y - \varepsilon \leqq F(f(y - \varepsilon) + 0) \leqq F(f(y - 0) + 0).$$

as ε may be chosen arbitrarily small, we have $y \leqq F(f(y - 0) + 0)$, hence $f(y) \leqq f(y - 0)$. (iii) : Case when $F(x) = 1$ for some x. It is the same as with (ii). Case when $F(x) < 1$ for all x. Given any number K, if $F(K + 0) < y < 1$, then $F(K + 0) < y \leqq (f(y) + 0)$, hence $f(y) > K$. Therefore we have $f(1 - 0) = \infty = f(1)$.

We call $f(y)$ *the inverse function* of the $F(x)$.

From the definition of $f(y)$, it is seen that

(A) $F(f(y) + 0) \geqq y$,

(B) $F(x + 0) \geqq y \longrightarrow x \geqq f(y)$.

$P \longrightarrow Q$ means that if P then Q,

Lemma 1.

(B′) $x < f(y) \longrightarrow F(x + 0) < y$,

(C) $x \geqq f(y) \longrightarrow F(x + 0) \geqq y$,

(D) $x > f(y + 0) \longrightarrow F(x) > y$,

(E) $x \leqq f(y + 0) \longrightarrow F(x) \leqq y$,

(E′) $y < F(x) \longrightarrow f(y + 0) < x$,

(D′) $y \geqq F(x) \longrightarrow f(y + 0) \geqq x$,

(C′) $y > F(x + 0) \longrightarrow f(y) > x$,

(B) $y \leqq F(x + 0) \longrightarrow f(y) \leqq x$,

Proof. (C) : $F(x + 0) \geqq F(f(y) + 0) \geqq y$. (D) : If $x > f(y + 0)$, we can choose $y' > y$ such that $x > f(y')$. $F(x) \geqq F(f(y') + 0) \geqq y' > y$. (E) : If $x \leqq f(y + 0)$, for any $\varepsilon > 0$, we have $x - \varepsilon < f(y + \varepsilon)$. Using (B′), $F(x - \varepsilon) < y + \varepsilon$. Since ε may be arbitrarily small, we have $F(x - 0) \leqq y$. (B′), (C′), (D′), (E′) are contrapositions of (B), (C), (D), (E) respectively.

From (E) and (D′) we have the following

Theorem 1. A d.f. $F(x)$ is uniquely determined by its inverse function $f(y)$. More explicitly,

$F(x) = \min \{y ; f(y + 0) \geqq x,\ 0 \leqq y \leqq 1\}$.

From (D′), (B), (C) and (E) we have the following

Lemma 2. $F(x - 0) \leqq y \leqq F(x + 0) \rightleftarrows f(y - 0) \leqq x \leqq f(y + 0)$.

$P \rightleftarrows Q$ means that if P then Q and conversely.

From (D′), (E′), (C), (B′) and simple conciderations we have the following

Lemma 3. $F(x_1) < y < F(x_2) \longrightarrow x_1 \leqq f(y) \leqq x_2,$

$f(y_1) < x < f(y_2) \longrightarrow y_1 \leqq F(x) \leqq y_2.$

By the *graph* Γ_F of $F(x)$, we mean the set

$\{(x, y) ; F(x-0) \leqq y \leqq F(x+0), -\infty < x < \infty\}$

and by the *graph* Γ_f, of $f(y)$, the set

$\{(x, y) : f(y-0) \leqq x \leqq f(y+0), -\infty < x < \infty, 0 \leqq y \leqq 1\},$

These graphs are easily proved to be a continuous curve, by the rotation of the co-ordinate axes. From Lemma 2, it is seen that the graph Γ_F of a d. f. $F(x)$ and the graph Γ_f of its inverse function $f(y)$ are coincident.

From Lemma 2, we have

$$\begin{aligned} f(y-0) &= \min\{x ; F(x-0) \leqq y \leqq F(x+0)\}, \ (0<y<1), \\ f(y+0) &= \max\{x ; F(x-0) \leqq y \leqq F(x+0)\}, \ (0<y<1), \\ F(x-0) &= \min\{y ; f(y-0) \leqq x \leqq f(y+0)\}, \\ F(x+0) &= \max\{y ; f(y-0) \leqq x \leqq f(y+0)\}, \end{aligned} \tag{1}$$

Hence, if $f(p-0) \leqq \xi_p \leqq f(p+0)$, where $0<p<1$, then ξ_p is the quantile of order p of the d. f. $F(x)$, and conversely.

By the simple considerations, from (1), it is seen that

$$\begin{aligned} f(y-0) &= \min\{x ; F(x+0) \geqq y\}, \ (0<y<1), \\ f(y+0) &= \max\{x ; F(x-0) \leqq y\}, \ (0<y<1), \\ F(x-0) &= \min\{y ; f(y+0) \geqq x\}, \\ F(x+0) &= \max\{y ; f(y-0) \leqq x\}, \end{aligned} \tag{2}$$

Therefore, we have the following

Lemma 4. (i) *Given any y_0 such that $0 < y_0 < 1$, write $x_1 = f(y_0 - 0)$ and $x_2 = f(y_0 + 0)$. Then, for any positive ε, we have $F(x_1 - \varepsilon) < y_0 < F(x_2 + \varepsilon)$.* (ii) *Given any real number x_0, write $y_1 = F(x_0 - 0)$ and $y_2 = F(x_0 + 0)$. Then, for any positive ε, we have $f(y_1 - \varepsilon) < x_0 < f(y_2 + \varepsilon)$.*

Theorem 2. If we write $f(y)$ the inverse function of a d.f. $F(x)$, then the inverse function of $F(ax+b)$, where $a>0$, is equal to $\{f(y)-b\}/a$.

Proof. Write $F_1(x) = F(ax+b)$, then

$$F_1\left(\frac{f(y)-b}{a} + 0\right) = F(f(y)+0) \geqq y.$$

Hence,

$$f_1(y) \leqq \frac{f(y)-b}{a}, \tag{3}$$

where $f_1(y)$ denotes the inverse function of $F_1(x)$.

As

$$F(af_1(y) + b + 0) = F_1(f_1(y) + 0) \geqq y,$$

we have

$$f(y) \leqq af_1(y) + b. \tag{4}$$

From (3) and (4), we have

$$f_1(y) = \frac{f(y) - b}{a}.$$

Theorem 3. Let $\{F_n(x)\ ;\ n = 0, 1, 2, \ldots\ldots\}$ be a sequence of d.f.'s, and $\{f_n(y)\}$ the corresponding sequence of inverse functions. A necessary and sufficient condition for the convergence of $\{F_n(x)\ ;\ n = 1, 2, \ldots\ldots\}$ to $F_0(x)$, in every continuity point of the latter, is that $\{f_n(y)\ ;\ n = 1, 2, \ldots\ldots\}$ converges to $f_0(y)$ in every continuity point of the latter.

The necessity is known. We can find it, for example, in [3], without proof. The sufficiency is probably known. We will give a proof for the completeness.

Proof. To prove the necessity, suppose that $\{F_n(x)\}$ converges to $F_0(x)$ in every continuity point of $F_0(x)$, and that $0 < y_0 < 1$. Write

$$x_1 = f_0(y_0 - 0), \qquad x_2 = f_0(y_0 + 0).$$

Given any positive ε, using Lemma 4, we have

$$F_0(x_1 - \varepsilon) < y_0 < F_0(x_2 + \varepsilon)$$

Let $x_1 - \varepsilon$ and $x_2 + \varepsilon$ be continuity points of $F_0(x)$, then by the assumption, $\{F_n(x_1 - \varepsilon)\}$ and $\{F_n(x_2 + \varepsilon)\}$ converge to $F_0(x_1 - \varepsilon)$ and $F_0(x_2 + \varepsilon)$ respectively.

Hence, we can choose N such that

$$F_n(x_1 - \varepsilon) < y_0 < F_n(x_2 + \varepsilon)$$

for all $n \geqq N$. From Lemma 3, we have

$$x_1 - \varepsilon \leqq f_n(y_0) \leqq x_2 + \varepsilon, \qquad n \leqq N.$$

Since ε may be chosen arbitrarily small, $\{f_n(y_0)\}$ converges to $f_0(y_0)$, assuming that y_0 is a continuity point of $f_0(y)$.

Similarly, we can prove tne sufficiency.

It is seen that the convergence at a point of the sequence of the inverse functions is a local property of the sequence of d. f.'s. That is:

Corollary 1. Let b be a continuity point of $f_0(y)$ and $a = f(b)$. In order that $\{f_n(b)\}$ converges to $f_0(b)$, it is sufficient that there exists a positive number ε such that if $a - \varepsilon < x < a + \varepsilon$ and if x is continuity point of $F_0(x)$, then $F_n(x)$ converges to $F_0(x)$.

Corollary 2. Let $F_n(x)$ be a sequence of d.f.'s, let p be a given number such that $0 < p < 1$, and let ξ_n be any quantile of order p of $F_n(x)$ for each n. If $\{F_n(x)\}$ converges to $F_0(x)$ in every continuity point of $F_0(x)$, then

$\{\xi_n\}$ is bounded and its any accumulating value is a quantile of order p of $F_0(x)$. In particular, if the quantile of order p of $F_0(x)$ is uniquely determined then $\lim_{n\to\infty}\xi_n = \xi_0$.

We can find the Slutsky's proof of Corollany 2 for the case when $p=\frac{1}{2}$ in [4], Chapt. VI, § 43.

Using the P. Lévy's metrization for the convergence in law, the Theorem 3 is the concequence of the following three Lemmas.

Lemma 5. A necessary and sufficient condition for the convergence of $\{F_n(x)\}$ to $F_0(x)$, at every continuity point of the latter, is that for any positive number ε, we can choose $N=N(\varepsilon)$ such that

$$F_0(x-\varepsilon)-\varepsilon \leqq F_n(x) \leqq F_0(x+\varepsilon)+\varepsilon, \quad (-\infty < x < \infty) \tag{5}$$

for all $n \geqq N$. (cf. P. Lévy [4], Chapt. III, § 17.)

Lemma 6. A necessary and sufficient condition for the convergence of $\{f_n(y)\}$ to $f_0(y)$, at every continuity point of the latter, is that for any positive number ε, we can choose $N=N(\varepsilon)$ such that

$$f_0(y-\varepsilon)-\varepsilon \leqq f_n(y) \leqq f_0(y+\varepsilon)+\varepsilon, \quad (0 \leqq y \leqq 1) \tag{6}$$

for all $n \geqq N$, assuming that

$$f_0(y) = \begin{cases} +\infty, & \text{if } y>1, \\ -\infty, & \text{if } y<0. \end{cases}$$

In (6), $(0 \leqq y \leqq 1)$ can be replaced by $(0<y<1)$.

Lemma 7. Let $F(x)$, $G(x)$ be d.f.'s and $f(y)$, $g(y)$ be the corresponding inverse functions, then the following two conditions (7) and (8) are equivalent:

$$F(x-\varepsilon)-\varepsilon \leqq G(x) \leqq F(x+\varepsilon)+\varepsilon, \quad (-\infty < x < \infty), \tag{7}$$

$$f(y-\varepsilon)-\varepsilon \leqq g(y) \leqq f(y+\varepsilon)+\varepsilon, \quad (0 \leqq y \leqq 1), \tag{8}$$

where

$$f(y) = \begin{cases} +\infty, & \text{if } y>1, \\ -\infty, & \text{if } y<0. \end{cases}$$

Proof of Lemma 7. As the function $F(x)$, $G(x)$, $f(y)$ and $g(y)$ are all everywhere continuous to the left, (7) and (8) are equivalent to the following (7′) and (8′), respectively:

$$F(x-\varepsilon-0)-\varepsilon \leqq G(x-0) \leqq G(x+0) \leqq F(x+\varepsilon+0)+\varepsilon, \quad (-\infty<x<\infty), \tag{7′}$$

$$f(y-\varepsilon-0)-\varepsilon \leqq g(y-0) \leqq g(y+0) \leqq f(y+\varepsilon+0)+\varepsilon, \quad (0 \leqq y \leqq 1), \tag{8′}$$

(7′) ((8′)) is equivalent to the condition that the graph Γ_G (Γ_g) of $G(x)$

$(g(y))$ lies between two curves obtained by translating the graph Γ_F (Γ_f) of $F(x)$ $(f(y))$ in the direction of the straight line $x+y=0$ by $\sqrt{2}\varepsilon$. Since $\Gamma_G=\Gamma_g$ and $\Gamma_F=\Gamma_f$, (7′) and (8′) are equivalent.

§ 2. The convergence of classes of d. f. 's. Let $F(x)$ be a d. f. A point x such that $F(x+\varepsilon)-F(x-\varepsilon)>0$ for all $\varepsilon>0$, is called a *increasing point* of $F(x)$. If $F(x)$ has at least two increasing points, then $F(x)$ is called to be *proper*, and otherwise, *unproper*. It is easily seen that denoting $f(y)$ the inverse function of $F(x)$, $f(y)$ is constant or not in $0<y<1$, according as $F(x)$ is unproper or proper.

Now, we write the uniqueness theorem of the limiting proper class of a sequence of classes of d. f. 's, in the following form.

Theorem 4. Let $\{F_n(x)\,;\ n=1,2,\ldots\ldots\}$ be a sequence of d.f.'s. Assume that there exist sequences of positive numbers $\{a_n\}$, $\{\alpha_n\}$, and sequences of real numbers $\{b_n\}$, $\{\beta_n\}$; and that there exist proper d.f.'s $\Phi(x)$, $\Psi(x)$ such that

$$\lim_{n\to\infty} F_n(a_nx+b_n)=\Phi(x), \qquad (a_n>0), \tag{9}$$

$$\lim_{n\to\infty} F_n(\alpha_nx+\beta_n)=\Psi(x), \qquad (\alpha_n>0), \tag{10}$$

at every continuity point of $\Phi(x)$ or $\Psi(x)$, respectively. Then there exist the limits

$$\lim_{n\to\infty}\frac{\alpha_n}{a_n}=A>0, \qquad \lim_{n\to\infty}\frac{\beta_n-b_n}{a_n}=B, \tag{11}$$

and for all x

$$\Psi(x)=\Phi(Ax+B), \tag{12}$$

That is, $\Phi(x)$ and $\Phi(x)$ belong to the same class.

Proof. Let $f_n(y)$, $\varphi(y)$, $\psi(y)$ be the inverse functions of the d. f. 's $F_n(x)$, $\Phi(x)$, $\Psi(x)$ respectively. From Theorem 2, the inverse functions of $F_n(a_nx+b_n)$, $F_n(\alpha_nx+\beta_n)$ are given by $\{f_n(y)-b_n\}/a_n$, $\{f_n(y)-\beta_n\}/\alpha_n$ respectively. From (9), (10), using Theorem 3, we have

$$\lim\,\{f_n(y)-b_n\}/a_n=\varphi(y), \tag{13}$$

$$\lim\,\{f_n(y)-\beta_n\}/\alpha_n=\psi(y), \tag{14}$$

at every continuity point of $\varphi(y)$ or $\psi(y)$, respectively. By the assumption, $\Phi(x)$, $\Psi(x)$ are proper d. f. 's, therefore, neither $\varphi(y)$ nor $\psi(y)$ is constant. We may find y_1, y_2 such that $1>y_1>y_2>0$, $\varphi(y_1)>\varphi(y_2)$, $\psi(y_1)>\psi(y_2)$; and that y_1, y_2 are both common continuity points of $\varphi(y)$ and $\psi(y)$. For $y=y_1$, y_2, we have (13) and (14). By making differences, we have

$$\lim\,\{f_n(y_1)-f_n(y_2)\}/a_n=\varphi(y_1)-\varphi(y_2)>0,$$

and

$$\lim \{f_n(y_1) - f_n(y_2)\}/\alpha_n = \psi(y_1) - \psi(y_2) > 0.$$

By concidering the ratio, we have

$$\lim \frac{\alpha_n}{a_n} = \frac{\varphi(y_1) - \varphi(y_2)}{\psi(y_1) - \psi(y_2)} \ (= A \text{ say}) > 0. \qquad (15)$$

From (14) and (15), we have

$$\lim \frac{f_n(y) - \beta_n}{a_n} = A\psi(y). \qquad (16)$$

By making (13) — (16), we have

$$\lim \frac{\beta_n - b_n}{a_n} = \varphi(y) - A\psi(y) \ (= B \text{ say}).$$

The last is true for any y which is comman continuity point of $\varphi(y)$ and $\psi(y)$. Since $\varphi(y)$ and $\psi(y)$ are both continuous to the left, we have for all y in $0 < y < 1$,

$$\varphi(y) - A\psi(y) = B.$$

$$\therefore \quad \psi(y) = \frac{\varphi(y) - B}{A}, \qquad (0 < y < 1).$$

Thus $\Psi(x)$ and $\Phi(Ax+B)$ have the same inverse function. By Theorem 1, we have (12).

Corollary 1. Assume that

$$\lim F_n(a_n x + b_n) = \Phi(x), \qquad (a_n > 0)$$

and

$$\lim F_n(\alpha_n x + \beta_n) = \Phi(x), \qquad (\alpha_n > 0),$$

in every continuity point of $\Phi(x)$; and that $\Phi(x)$ is a proper d.f. Then we have

$$\lim \frac{\alpha_n}{a_n} = 1, \qquad \lim \frac{\beta_n - b_n}{a_n} = 0.$$

(*cf.* [1])

Proof. From the above theorem, we have

$$\Phi(x) = \Phi(Ax + B), \qquad (-\infty < x < \infty),$$

where A and B are defined by (11). Using Theorem 2, we have

$$\varphi(y) = (\varphi(y) - B)/A, \qquad (0 < y < 1),$$

where $\varphi(y)$ denotes the inverse function of $\Phi(x)$. Since $\Phi(x)$ is proper, $\varphi(y)$ may assume at least two different values, therefore,

$$A = 1, \qquad B = 0.$$

Corollary 2. Assume that

$$\lim F_n(a_n x + b_n) = \Phi(x), \qquad (a_n > 0)$$

and

$$\lim F_n(\alpha_n x + \beta_n) = \Psi(x), \qquad (\alpha_n > 0),$$

at every continuity point of $\Phi(x)$ *and* $\Psi(x)$, *respectively, and assume that* $\Phi(x)$ *is unproper and* $\Psi(x)$ *is proper. Then there exist the limits*

$$\lim \frac{\alpha_n}{a_n} = 0, \qquad \lim \frac{\beta_n - b_n}{a_n} = B,$$

and

$$\Phi(x) = \begin{cases} 0, & x < B, \\ 1, & x > B. \end{cases} \tag{17}$$

Note that (12) and (17) may be rewritten in the following same expression

$$\Phi(x) = \lim_{A' \downarrow A} \Phi\left(\frac{x - B}{A'}\right)$$

Corollary 3. Assume that

$$\lim F_n(a_n x + b_n) = \Phi(x), \qquad (a_n > 0),$$

at every continuity point of $\Phi(x)$ *and that there exist the limits*

$$\lim_{n\to\infty} \frac{\alpha_n}{a_n} = A \quad (\alpha_n > 0), \qquad \lim \frac{\beta_n - b_n}{a_n} = B,$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$, $\{b_n\}$ *are sequences of constants. (It is indifferent that* $\Phi(x)$ *is proper or not). If* $A > 0$, *then*

$$\lim F_n(\alpha_n x + \beta_n) = \Phi(Ax + B).$$

If $A = 0$, *then*

$$\Phi(B - 0) \leqq \liminf_{n\to\infty} F_n(\alpha_n x + \beta_n) \leqq \limsup_{n\to\infty} F_n(\alpha_n x + \beta_n) \leqq \Phi(B + 0).$$

Proof. Case when $A > 0$.

$$\lim \frac{f_n(y) - \beta_n}{\alpha_n} = \lim \left(\frac{f_n(y) - b_n}{a_n} - \frac{\beta_n - b_n}{a_n}\right) \Big/ \frac{\alpha_n}{a_n} = \frac{\varphi(y) - B}{A}.$$

By Theorem 3, we have the desired result.

Case when $A = 0$. It is clear from the following fact. Let b be any number such that $0 < b < 1$. If for all $y > b$, $\lim f_n(y) = +\infty$, then we have $\limsup F_n(x) \leqq b$ for any x. If for all $y < b$, $\lim f_n(y) = -\infty$, then we have $\liminf F_n(x) \geqq b$ for any x.

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