On a Relation between Exponential Law and Poisson's Law

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Let

$$(1) \qquad X_1, X_2, \cdots$$

be a sequence of non-negative random variables independently distributed according to the same distribution function $F(x)$. Let x be any positive number, and N_x be a random variable defined as follows:

$$(2) \qquad N_x = \begin{cases} 0 & \text{if } X_1 > x, \\ n & \text{if } X_1 + X_2 + \cdots + X_n \leqq x \\ & \quad \text{and } X_1 + X_2 + \cdots + X_{n+1} > x, \end{cases}$$

that is, N_x be the least non-negative integer such that $X_1 + X_2 + \cdots\cdots + X_{N_x+1} > x$.

If $F(x)$ is the exponential distribution function with the mean μ, *i. e.*,

$$(3) \qquad F(x) = \begin{cases} 0 & \text{for } x < 0, \\ \dfrac{1}{\mu}\displaystyle\int_0^x e^{-\frac{x}{\mu}}\,dx = 1 - e^{-\frac{x}{\mu}} & \text{for } x > 0, \end{cases}$$

then N_x is, as is known, distributed according to the Poisson's law with the mean x/μ.

In this paper we shall prove the converse of this fact.[*)]

Theorem. *Let* (1) *be a sequence of non-negative independent random variables with the same distribution function* $F(x)$, *and let* N_x *be defined as* (2). *If* N_x *is distributed according to the Poisson's law for every positive* x, *then* $F(x)$ *is of exponential type* (3).

Proof. First, the mean of N_x, existing by the assumption, is clearly a function defined for all $x > 0$, which we denote by $f(x)$. Then it follows from the definition of N_x

$$(4) \qquad P_r\{X_1 + X_2 + \cdots + X_n \leqq x\} = P_r\{N_x \geqq n\}$$
$$= 1 - e^{-f(x)}\left\{1 + \frac{f(x)}{1\,!} + \frac{f^2(x)}{2\,!} + \cdots + \frac{f^{n-1}(x)}{(n-1)\,!}\right\}.$$

Denoting the distribution function of $X_1 + X_2 + \cdots\cdots + X_n$ by $F_n(x)$, we have from (4)

*) This problem was presented to the author by Mr. J. Ogawa.

$$(5)\quad F_n(x)=\begin{cases}0 & \text{for}\quad x<0,\\ 1-e^{-f(x)}\left\{1+\dfrac{f(x)}{1!}+\dfrac{f^2(x)}{2!}+\cdots+\dfrac{f^{n-1}(x)}{(n-1)!}\right\} & \text{for}\quad x>0.\end{cases}$$

In particular,

$$(6)\quad F(x)=F_1(x)=\begin{cases}0 & \text{for}\quad x<0,\\ 1-e^{-f(x)} & \text{for}\quad x>0.\end{cases}$$

Now, $F(x)$ being non-decreasing and $F(+\infty)=1$, so $f(x)$ is also non-decreasing and

$$(7)\quad f(+\infty)=+\infty.$$

On the other hand, since

$$P_r\{X_1+X_2=0\}=P_r\{X_1=0\}P_r\{X_2=0\},$$

we have

$$F_2(+0)=F^2(+0),$$

$$1-e^{-f(+0)}\{1+f(+0)\}=\{1-e^{-f(+0)}\}^2,$$

from which it follows

$$(8)\quad f(+0)=0.$$

Let $\varphi_n(t)$ be the moment-generating function corresponding to $F_n(x)$. Calculating the value of $\varphi_n(-t)$ for a fixed negative argument $-t$, where $t>0$, and taking into account of (5), (7) and (8), we have

$$(9)\quad \begin{aligned}\varphi_n(-t)&=\int_0^\infty e^{-tx}\,dF_n(x)\\ &=\Big[e^{-tx}F_n(x)\Big]_0^\infty+t\int_0^\infty e^{-tx}F_n(x)\,dx\\ &=t\int_0^\infty e^{-tx}F_n(x)\,dx\\ &=1-t\int_0^\infty e^{-tx-f(x)}(x)\sum_{k=0}^{n-1}\frac{f^k(x)}{k!}\,dx.\end{aligned}$$

In particular, putting $\varphi(-t)=\varphi_1(-t)$, for $n=1$,

$$(10)\quad \begin{aligned}&\varphi(-t)=\varphi_1(-t)=1-t\int_0^\infty e^{-tx-f(x)}\,dx,\\ &0<\varphi(-t)<1,\end{aligned}$$

and

$$(11)\quad \varphi_n(-t)=\varphi^n(-t).$$

Then, from (9) and (11) it follows

$$\varphi^n(-t) = 1 - t\int_0^\infty e^{-tx-f(x)} \sum_{k=0}^{n-1} \frac{f^k(x)}{k!}\, dx. \tag{12}$$

Now, let u be an arbitrary complex number such that $|u| < 1$. By multiplying both sides of (12) by u^{n-1} for $n = 1, 2, \cdots\cdots$, and adding the obtained results side by side, we have

$$\frac{\varphi(-t)}{1-u\varphi(-t)} = \frac{1}{1-u} - \frac{t}{1-u}\int_0^\infty e^{-tx-f(x)+uf(x)}\, dx, \tag{13}$$

$$\int_0^\infty \frac{t}{1-\varphi(-t)} e^{-tx-f(x)+uf(x)}\, dx = \frac{1}{1-u\varphi(-t)}.$$

In this last expression put $u = 0$ and we get

$$\int_0^\infty g(x)\, dx = 1,$$

where

$$g(x) = \frac{t}{1-\varphi(-t)} e^{-tx-f(x)} > 0 \qquad \text{for} \quad x \geqq 0. \tag{14}$$

Hence, we can regard $g(x)$ as the probability density of a non-negative random variable, say X. Then (13) can be rewritten in the form

$$\int_0^\infty g(x)\, e^{uf(x)}\, dx = \frac{1}{1-u\varphi(-t)}. \tag{15}$$

The left hand side of this equality is the moment-generating function of $f(X)$, while the right hand side is that of the exponential distribution with the mean $\varphi(-t)$. Therefore, $f(X)$ must be distributed according to this law, and $f(x)$ takes all positive values when x runs over all positive values, that is, the function $f(x)$ is free from discontinuities.

On the other hand we have $f(x) < f(x')$ for $x < x'$. Otherwise, we should have from $f(x) = f(x')$ and (14)

$$P_r\{f(X) = f(x)\} \geqq P_r\{x \leqq X \leqq x'\} = \int_x^{x'} g(x)\, dx > 0,$$

which is impossible because $f(X)$ is a random variable with a continuous distribution function.

Therefore, the mapping $x \rightleftarrows f(x)$ defined on all positive values is one-to-one and continuous. Moreover $f(x)$ is an absolutely continuous function of x. In fact, if E is a Borel set of Lebesgue measure 0, i. e., $m(E) = 0$, then we have from the above established one-to-one correspondence

$$0 = \int_E g(x)\, dx = \frac{1}{\varphi(-t)}\int_{f(E)} e^{-\frac{x}{\varphi(-t)}}\, dx,$$

which is impossible unless $m(f(E))=0$.

Now, since

$$P_r\{X\leqq x\}=P_r\{f(X)\leqq f(x)\},$$

we have for $x>0$,

$$\int_0^x g(x)\,dx=\frac{1}{\varphi(-t)}\int_0^{f(x)} e^{-\frac{x}{\varphi(-t)}}\,dx. \tag{16}$$

Differentiating both sides of (16) with respect to x, we obtain

$$g(x)=\frac{1}{\varphi(-t)}\,e^{-\frac{f(x)}{\varphi(-t)}}f'(x) \tag{17}$$

almost everywhere. From (14) and (17) we have

$$te^{-tx}=\frac{1-\varphi(-t)}{\varphi(-t)}\,e^{-\frac{1-\varphi(-t)}{\varphi(-t)}f(x)}f'(x). \tag{18}$$

Integrating both sides of (18) with respect to x from 0 to x, we have for $x>0$

$$1-e^{-tx}=1-e^{-\frac{1-\varphi(-t)}{\varphi(-t)}f(x)},$$

from which we get

$$f(x)=\frac{t\varphi(-t)}{1-\varphi(-t)}\,x=\frac{x}{\mu} \tag{19}$$

where

$$\mu=\frac{1-\varphi(-t)}{t\varphi(-t)}.$$

Substituting (19) in (6), we have the desired result.

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