On the Fundamental Operations of Collectives

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1. As is known, R. v. Mises constructed the theory of probability from the standpoint of collectives and later, A. Wald made the important contribution to it.*) Wald gave the solution to the following problem:

Let M be a label space. What are the conditions, that the system $\mathfrak{S}$ of selection rules, the system $\mathfrak{F}$ of subsets of M, and the set function μ defined on $\mathfrak{F}$ satisfy, in order that a collective $K(\mathfrak{S}, \mathfrak{F})$ exist, whose distribution function is identical with μ?

From his article one can easily learn the importance of the rôle of $\mathfrak{S}$ in the calculus of collectives. In this note I shall show what selection rules $\mathfrak{S}$ must contain, in order that some fundamental operations are possible, and what system of selection rules are admitted by the resulted collectives.

2. First, we begin with the four fundamental operations.

I. Place selection. This is an application of a selection rule to a collective and is the simplest operation. Let $K(\mathfrak{S}, \mathfrak{F})$ be a collective with regard to the system $\mathfrak{S}$ of selection rules and the system $\mathfrak{F}$ of subsets of M, and let f be an arbitrary element of $\mathfrak{S}$. Then, a necessary and sufficient condition that $f(K(\mathfrak{S}, \mathfrak{F}))$ be also a collective with regard to $\mathfrak{S}$ and $\mathfrak{F}$, is that $\mathfrak{S}$ is a semi-group (with unit) concerning the product of selection rules, or that the semi-group generated by the elements of $\mathfrak{S}$ are admissible by $K(\mathfrak{S}, \mathfrak{F})$.

II. Mixing. Let A and B be two disjoint sets of $\mathfrak{F}$. Then, identifying the elements of A and B is a mixing operation on $K = K(\mathfrak{S}, \mathfrak{F})$. Let M', $\mathfrak{F}'$ and K' be the label space, the system of subsets of M', and the sequence of elements of M' obtained from M, $\mathfrak{F}$ and K by this identification respectively. We denote the correspondence from M onto M' by $m' = \varphi(m)$, where to an element not belonging to A or B does itself correspond and to an element belonging to A or B the same element of M', say $\overline{m}'$.

If we put

$$f_{\varphi 0} = f_0,$$

*) A. Wald, *Widerspruchsfreiheit des Kollektivbegriffes der Wahrscheinlichkeitrechnung*, Ergebnisse eines mathematischen Kolloquiums Wien, 1937.

$$f_{\varphi n}\big(\varphi(m_1), \varphi(m_2), \cdots, \varphi(m_n)\big) = f_n(m_1, m_2, \cdots, m_n) \quad (n = 1, 2, \cdots)$$

for $f = \{f_n\} \in \mathfrak{S}$ and an arbitrary sequence $\{m_i\}$ in M, then $f_\varphi = \{f_{\varphi, n}\}$ defines a place selection for $\{\varphi(m_i)\}$. Conversely, if we put

$$f_0 = f_0',$$

$$f_n(m_1, \cdots, m_n) = f_n'\big(\varphi(m_1), \cdots, \varphi(m_n)\big) \quad (n=1, 2, \cdots),$$

for an arbitrary sequence $\{m_n\}$ in M, then $f = \{f_n\}$ is obviously a place selection for $\{m_n\}$. In this case, however, $\{m_n\}$ being an arbitrary sequence in M, $f = \{f_n\}$ defines a selection rule in M. The correspondence between f and f' will be denoted by $f = \Psi_{M' \to M}(f')$. We have then

$$\big(\Psi_{M' \to M}(f')\big)_\varphi = f.$$

Now, setting

$$\mathfrak{S}' = \big\{f\,;\ \Psi_{M' \to M}(f) \in \mathfrak{S}\big\}.$$

K' is a collective with regard to $\mathfrak{S}'$ and $\mathfrak{F}'$. We have clearly

$$P_{K'}(\overline{m}') = P_K(A) + P_K(B),$$

where $P_{K'}(\overline{m}')$, $P_K(A)$ and $P_K(B)$ represent the probability of $(\overline{m}')$ in K', those of A and B in K, respectively.

III. Partition. Assume that M contains more than two elements, and let A be an element of $\mathfrak{F}$ with $P_K(A) \neq 0$. When we select the elements belonging to A from K, we have a sequence K' consisting of elements in A. Selecting K' out of K in this way is called partition, as is known. We denote this operation by $K' = T_A(K)$. Then, if B is a subset contained in A and belonging to $\mathfrak{F}$, the relative frequency of B in K' has the limit $P_K(B) / P_K(A)$.

Now, let $\{p_n\}$ be an arbitrary sequence in M', and let $g = \{g_n\}$ be a selection rule defined in A. Then, setting

$$f_0 = g_0,$$

$$f_n(p_1, \cdots, p_n) = g_k(p_{i_1}, \cdots, p_{i_k}) \qquad (n = 1, 2, \cdots),$$

where $p_{i_1}, \cdots, p_{i_k}$ are elements among $p_1, \cdots, p_n$ belonging to A and $i_1 < i_2 < \cdots < i_k$, we have a place selection $f = \{f_n\}$ for $\{p_n\}$. Since $\{p_n\}$ is an arbitrary sequence in M, $f = \{f_n\}$ defines a selection rule in M. The correspondence from g to f will be denoted by $f = \Phi_{A \to M}(g)$. For an arbitrary sequence K_1 in M it obviously holds

$$T_A\big\{f(K_1)\big\} = g\big(T_A(K_1)\big).$$

If $f \in \mathfrak{S}$, the relative frequencies of A and B in $f(K)$ have the limit

$P_K(A)$ and $P_K(B)$, respectively. Consequently, the limit of the relative frequency of B in $g(K')$ is $P_K(B)/P_K(A)$. Setting

$$\mathfrak{S}' = \left\{g\,;\,\Psi_{A\to M}(g) \in \mathfrak{S}\right\},$$

K' admits all the elements of $\mathfrak{S}'$, and if $\mathfrak{S}$ is enumerable, so is also $\mathfrak{S}'$. Since $\mathfrak{S}'$ contains the unit selection rule, it is not empty. Denoting by $\mathfrak{F}'$ the system of subsets which belong to $\mathfrak{F}$ and are contained in A, $K' = T_A(K)$ is a collective with regard to $\mathfrak{S}'$ and $\mathfrak{F}'$. The probability of B of $\mathfrak{F}'$ in K' is given by

$$P_{K'}(B) = \frac{P_K(B)}{P_K(A)}.$$

IV. Combination. Let $K_1 = K_1(\mathfrak{S}_1, \mathfrak{F}_1) = \{m_i^{(1)}\}$, $K_2 = K_2(\mathfrak{S}_2, \mathfrak{F}_2) = \{m_i^{(2)}\}$ be two collectives in the label spaces $M^{(1)}$ and $M^{(2)}$ respectively. Then, the sequence $K = \left\{(m_i^{(1)}, m_i^{(2)})\right\}$ is a sequence in the product space $M^{(1)} \times M^{(2)}$. If K is a collective with regards to some system $\mathfrak{S}$ of selection rules and $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$ in $M^{(1)} \times M^{(2)}$, K_1 and K_2 are called combinable and the operation to form K from K_1 and K_2 is called combination.

Now, assume that K_1 and K_2 are combinable. Further, let $n_{A^{(1)}}$ be the number of elements in the first n terms of K_1 which belong to $A^{(1)}$ of $\mathfrak{F}_2$, and let $n_{A^{(1)}\times A^{(2)}}$ be the number of elements in the first n terms of K which belong to $A^{(1)} \times A^{(2)}$, where $A^{(2)}$ is an element of $\mathfrak{F}_2$. Then we have

$$(*) \qquad \frac{n_{A^{(1)}\times A^{(2)}}}{n} = \frac{n_{A^{(1)}}}{n} \cdot \frac{n_{A^{(1)}\times A^{(2)}}}{n_{A^{(1)}}}.$$

As $n \to \infty$, the left-hand side of this identity $(*)$ and $n_{A^{(1)}}/n$ in the right-hand side have the limit, therefore, $n_{A^{(1)}\times A^{(2)}}/n_{A^{(1)}}$ must have the limit. Of course, we have assumed $P_{K_1}(A^{(1)}) \neq 0$. $n_{A^{(1)}\times A^{(2)}}/n_{A^{(1)}}$ is the relative frequency of $A^{(2)}$ in the first $n_{A^{(1)}}$ terms of the sequence $\{m_{i_j}^{(2)}\}$ $(j = 1, 2, \cdots)$, whose elements are those of K_2 such that the corresponding $m_{i_j}^{(1)}$ belong to $A^{(1)}$. To select $\{m_{i_j}^{(2)}\}$ from K_2 in this way is called $(K_1, A^{(1)})$-sampling. In the case, when the limit of $n_{A^{(1)}\times A^{(2)}}/n_{A^{(1)}}$ is equal to $P_{K_2}(A^{(2)})$, we have

$$(\ast\!\ast) \qquad P_K(A^{(1)} \times A^{(2)}) = P_{K_1}(A^{(1)})\,P_{K_2}(A^{(2)}).$$

Now, let $K_2' = \{m_{i_j}^{(2)}\}$ $(j = 1, 2, \cdots)$ be a subsequence of $K_2 = \{m_i^{(2)}\}$ selected for $f^{(1)} = \{f_n^{(2)}\}$ of $\mathfrak{S}_1$ in the following way:

each $m_{n+1}^{(2)}$ of K_2 is to be selected or not, according to

$$f_n^{(1)}(m_1^{(1)}, \cdots, m_n^{(1)}) = 1 \text{ or } 0.$$

We apply $\left(f^{(1)}(K_1), A^{(1)}\right)$-sampling to K_2' and denote by K_2'' the obtained sequence. If the limit of the relative frequency of each $A^{(2)}$ of $\mathfrak{F}_2$ in K_2''

is equal to the probability of $A^{(2)}$ in K_2, K_2 is called *independent* of K_1.

Setting for any $f^{(1)} = \{f_n^{(1)}\}$ out of $\mathfrak{S}$

$$f_0 = f_0^{(1)},$$

$$f_n\big((p_1^{(1)}, p_1^{(2)}), \cdots, (p_n^{(1)}, p_n^{(2)})\big) = f_n^{(1)}(p_1^{(1)}, \cdots, p_n^{(1)}) \qquad (n = 1, 2, \cdots),$$

where $\{p_n^{(1)}\}$, $\{p_n^{(2)}\}$ are arbitrary sequences in $M^{(1)}$ and $M^{(2)}$, respectively, $f = \{f_n\}$ defines obviously a selection rule in $M^{(1)} \times M^{(2)}$. This is called a selection rule induced by $f^{(1)}$ in $M^{(1)} \times M^{(2)}$. If $f^{(1)} \to f$, $g^{(1)} \to g$ in the correspondence between selection rules in $M^{(1)}$ and the induced ones in $M^{(1)} \times M^{(2)}$, then clearly $f^{(1)} \cdot g^{(1)} \to f \cdot g$.

In the case, when K_2 is independent of K_1, $K = \{m_i^{(1)}, m_i^{(2)})\}$ makes a collective with regard to the system of the selection rules induced in $M^{(1)} \times M^{(2)}$ by all selection rules of $\mathfrak{S}_1$ and $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$, and we have the relation (⁑). Therefore, when K_2 is independent of K_1, K_1 and K_2 are combinable.

It is to be noted that our definition of independence has not the property of symmetry, i. e., one cannot immediately conclude that K_1 is independent of K_2 even if K_2 is independent of K_1. However, the relation (⁑) holds in both cases, where K_2 is independent of K_1 or where K_1 is independent of K_2. The only defferent point is the difference of the systems of selection rules. If $K_1(\mathfrak{S}_1, \mathfrak{F}_1)$ and $K_2(\mathfrak{S}_2, \mathfrak{F}_2)$ are independent of each other, their combination admits all the selection rules induced in $M^{(1)} \times M^{(2)}$ by all $f^{(1)}$ of $\mathfrak{S}_1$ or all $f^{(2)}$ of $\mathfrak{S}_2$.

Now, let $K_1(\mathfrak{S}_1, \mathfrak{F}_1)$ and $K_2(\mathfrak{S}_2, \mathfrak{F}_2)$ satisfy the following condition:

(1) K_1 and K_2 are combinable,

(2) the combined collective is one with regard to the system $\mathfrak{S}$ of selection rules and $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$,

(3) the relation (⁑) holds.

Some selection rules in $M^{(1)} \times M^{(2)}$ are considered to define selection rules in $M^{(1)}$ or $M^{(2)}$. Then, if $\mathfrak{S}$ contains all selection rules induced by all elements of $\mathfrak{S}_1$, K_2 is independent of K_1. Moreover, if $\mathfrak{S}$ contains selection rules induced by all elements of $\mathfrak{S}_2$, K_1 and K_2 are, therefore, independent of each other.

3. In this section I shall state some important calculus.

Assume that we are given k collectives with the same system $\mathfrak{F}$ of subsets

$$K_1 = K_1(\mathfrak{S}_1, \mathfrak{F}) = \{m_n^{(1)}\}, \cdots, K_k = K_k(\mathfrak{S}_k, \mathfrak{F}) = \{m_n^{(k)}\}.$$

When we make the sequence

$$K = \{m_1^{(1)}, m_1^{(2)}, \cdots m_1^{(k)}, m_2^{(1)}, \cdots, m_2^{(k)}, m_3^{(1)}, \cdots\},$$

what collective does it make?

The relative frequency of an arbitrary element A of $\mathfrak{F}$ in K has obviously the limit

$$\frac{1}{k}\left\{P_{K_1}(A)+\cdots+P_{K_k}(A)\right\}.$$

Therefore, if $K_1, \cdots, K_k$ always admit $\left(f^{(1)}(K_1), M\right)$-sampling, $f^{(1)}$ being an arbitrary element of $\mathfrak{S}_1$, then K is a collective with regard to the system of selection rules induced by all the elements of $\mathfrak{S}_1$, and $\mathfrak{F}$. The simplest case is that where $K_1 = K_2 = \cdots = K_k$.

Now we shall consider the condition that $K_1, \cdots, K_k$ be combinable. As is easily seen, the following is a sufficient condition:

For an arbitrary element $f^{(1)}$ out of $\mathfrak{S}_1$ and k-1 arbitrary elements $A_1, \cdots, A_{k-1}$ of $\mathfrak{F}$

(1) K_2 admits $\left(f^{(1)}(K_1), A_1\right)$-sampling,

(2) Setting

$$\left(\left(f^{(1)}(K_1), A_1\right)(K_2), A_2\right) = \left(f^{(1)}(K_1), A_1;\, K_2, A_2\right),$$

K_3 admits $\left(f^{(1)}(K_1), A_1;\, K_2, A_2\right)$-sampling,

(3) Similarly, K_i admits $\left(f^{(1)}(K_1), A_1;\, K_2, A_2;\, \cdots, K_{i-1}, A_{i-1}\right)$-sampling $(i = 3, 4, \cdots, k)$.

Let us consider this condition in K.

For an arbitrary element $g^{(1)} = \{g_l^{(1)}\}$ $(l = 0, 1, 2, \cdots)$ of $\mathfrak{S}_1$ and arbitrary $A_1, \cdots, A_{k-1}$ of $\mathfrak{F}$ set

$$(1) \qquad g_{0,\, lk+j}(m_1^{(1)}, \cdots, m_1^{(k)}, m_2^{(1)}, \cdots, m_l^{(1)}, \cdots, m_l^{(k)}, m_{l+1}^{(1)}, \cdots, m_{l+1}^{(j)})$$
$$= \begin{cases} g_l^{(1)}(m_1^{(1)}, m_2^{(2)}, \cdots, m_l^{(1)}) & \text{if } j = 0, \\ 0 & \text{if } 1 \leqq j \leqq k-1 \quad (l = 0, 1, 2, \cdots), \end{cases}$$

$$(2) \qquad g_{\lambda,\, lk+j}(m_1^{(1)}, \cdots m_l^{(k)}, \cdots, m_l^{(1)}, \cdots, m_l^{(k)}, m_{l+1}^{(1)}, \cdots, m_{l+1}^{(j)})$$
$$= \begin{cases} 1 & \text{if } j = \lambda,\ m_{l+1}^{(1)} \text{ is selected by } g_{(0)} = \{g_{0,n}\}, \text{ and} \\ & \quad m_{l+1} \in A_1,\ m_{l+1}^{(2)} \in A_2, \cdots, m_{l+1}^{(j)} \in A_j, \\ 0 & \text{otherwise}, \end{cases}$$
$$(l = 0, 1, 2, \cdots).$$

$g_{(0)} = \{g_{0,n}\}, \cdots, g_{(k-1)} = \{g_{k-1,n}\}$ define selection rules in M.

Then, if K always admits such $g_{(0)}, g_{(1)}, \cdots, g_{(k-1)}, K_1, K_2, \cdots, K_k$ are combinable and for arbitrary $A_1, \cdots, A_k$, the probability of $A_1 \times \cdots \times A_k$ in the combined collective is

$$\frac{1}{k^k}\left(\sum_{i=1}^{k} P_{K_i}(A_1)\right)\cdots\left(\sum_{i=1}^{k} P_{K_i}(A_k)\right).$$

Especially, when $P_{K_1}(A_i) = \cdots = P_{K_k}(A_i)$ $(i = 1, 2, \cdots, k)$, it turns out to be $P_{K_1}(A_1) \cdots P_{K_k}(A_k)$. The combined collective admits the selection rules induced in $\underbrace{M \times \cdots \times M}_{k}$ by all elements of $\mathfrak{S}_1$.

For example, let $K_1, \cdots\cdots, K_k$ be subsequences of a collective $K = K(\mathfrak{S}, \mathfrak{F}) = \{m_n\}$ such as

$$(\ast_{\ast\ast}) \qquad \begin{aligned} K_1 &= \{m_{lk+1}\}, \quad (l = 0, 1, \cdots), \\ K_2 &= \{m_{lk+2}\}, \quad (l = 0, 1, \cdots), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ K_k &= \{m_{lk}\}, \quad (l = 1, 2, \cdots). \end{aligned}$$

In this case, if $K_1, K_2, \cdots\cdots, K_k$ are obtained by applying selection rules in $\mathfrak{S}$ to K, and if $\mathfrak{S}$ makes a semi-group, then $K_1, K_2, \cdots\cdots, K_k$ also are collectives with regard to $\mathfrak{S}$ and $\mathfrak{F}$, and the probability of any A of $\mathfrak{F}$ in each K_i is equal to that in K. Further, we can verify by the result of Wald the existence of a collective K such as the above-mentioned $K_2, \cdots\cdots, K_k$ are independent of K_1.

Let $\mathfrak{F}_0$ be a field containing at most countably many subsets in M, and let $\mu(A)$ be a non-negative and additive set function defined on $\mathfrak{F}_0$ with $\mu(M) = 1$. Further, let $\mathfrak{F}$ be a field of subsets measurable in Jordan sense concerning $\mathfrak{F}_0$ and μ, and let $\mathfrak{S}_0$ be a system of at most countably many selection rules in M. For every integer k the selection of the first term, the $(k+1)$-st term, the $(2k+1)$-st term, $\cdots\cdots$ out of a sequence in M will be denoted by a_k. For an arbitrary f of $\mathfrak{S}_0$ and arbitrary $A_1, \cdots\cdots, A_{k-1}$ of $\mathfrak{F}_0$ we consider the following selection rules $f^{(i)} = \{f_n^{(i)}\}$ $(i = 1, 2, \cdots, k-1)$,

$$(\ast\ast_{\ast\ast}) \qquad f_{lk+1}^{(i)}(m_1, \cdots, m_{lk+j}) = \begin{cases} 1 & \text{if } j = i,\ m_{lk+1} \text{ is selected by } fa_k, \text{ and} \\ & \quad m_{lk+1} \in A_1, \cdots, m_{lk+i} \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$
$$(l = 0, 1, 2, \cdots; j = 0, \cdots, k-1).$$

Such selection rules are at most enumerable, as a_k and system $(f, A_1, \cdots\cdots, A_{k-1})$ are enumerable. Let $\mathfrak{S}_0'$ be a semi-group of such rules, and let $\mathfrak{S}$ be a semi-group generated by $\mathfrak{S}_0$ and $\mathfrak{S}_0'$. Then $\mathfrak{S}$ is at most enumerable. Therefore, according to Wald's result there exist continuously many collectives with regard to $\mathfrak{S}$ and $\mathfrak{F}$, whose distributions are identical with $\mu(A)$.

Now, let $K = \{m_n\}$ be such a collective, and for an arbitrary integer k let $K_1, \cdots\cdots, K_k$ be such as $(\ast_{\ast\ast})$. Then $K_1, \cdots\cdots, K_k$ are collectives with regard to $\mathfrak{S}$ and $\mathfrak{F}_0$ and are combinable. The combined collective K^* is one with regard to the system $\mathfrak{S}^*$ of selection rules induced in $\underbrace{M \times \cdots \times M}_{k}$

by all elements of $\mathfrak{S}$ and $\underbrace{\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_0}_{k}$. For $A_1, \cdots, A_k \in \mathfrak{F}_0$

$$P_{K^*}(A_1 \times \cdots \times A_k) = P_K(A_1) \cdots P_K(A_k).$$

This collective K^* is proved to be a collective with regard to $\mathfrak{S}^*$ and $\underbrace{\mathfrak{F} \times \cdots \times \mathfrak{F}}_{k}$. Consequently, $K_1, K_2, \cdots, K_k$ also are those with regard to $\mathfrak{S}$ and $\mathfrak{F}$, and $K_2, \cdots, K_k$ are independent of K_1.

4. Conclusion. In order to develop the theory of probability from the standpoint of collectives the above-mentioned operations are inevitable. Therefore, it is necessary that the system $\mathfrak{F}$ of subsets is one as mentioned above, $\mathfrak{S}$ is a semi-group and that a collective $K(\mathfrak{S}, \mathfrak{F})$ admits the semi-group generated by $\mathfrak{S}$, all a_k, and selection rules as $\binom{**}{**}$.

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