The tail probability of the maximum of a bivariate Gaussian process
(2変量ガウス確率過程の最大値の裾確率)

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1 Euler characteristic method for a bivariate Gaussian process

Let $(x(s), y(t)) \in \mathbb{R}^2$, $(s,t) \in S \times T$, be a bivariate Gaussian process with a smooth sample path, where $S, T \subset \mathbb{R}^1$ are intervals. In this paper, we study the the Euler characteristic method for such a bivariate Gaussian process. The Euler characteristic method is based on the approximation

$$\mathbb{1}\left\{\sup_{s\in S} x(s) \ge a\right\} \approx \chi(S_a), \quad \mathbb{1}\left\{\sup_{t\in T} y(t) \ge b\right\} \approx \chi(T_b)$$

when a and b are large. Here S_a and T_b are the excursion sets defined by

$$S_a = \{s \in S \mid x(s) \ge a\}, \quad T_b = \{t \in T \mid y(t) \ge b\},$$

and $\chi(S_a)$ and $\chi(T_b)$ are the Euler characteristics (numbers of connected components) of S_a and T_b, respectively. If we admits this approximation, we would have

$$P\left(\sup_{s\in S} x(s) \ge a, \sup_{t\in T} y(t) \ge b\right) \approx E[\chi(S_a)\chi(T_b)].$$

We will evaluate $E[\chi(S_a)\chi(T_b)]$ when a and b are large.

(1-dimensional) Morse's theorem states that

$$\chi(S_a) = \sum_{s\in S^*\cap \mathrm{int}(S)} \mathbb{1}\{x(s) \ge a\}\mathrm{sgn}\bigl(-\ddot{x}(s)\bigr) + \sum_{s\in S^*\cap\partial S} \mathbb{1}\{x(s)\ge a\},$$
$$\chi(T_b) = \sum_{t\in T^*\cap \mathrm{int}(T)} \mathbb{1}\{y(t) \ge b\}\mathrm{sgn}\bigl(-\ddot{y}(t)\bigr) + \sum_{t\in T^*\cap\partial T} \mathbb{1}\{y(t)\ge b\},$$

where S^* and T^* are sets of augmented critical points of $x(s)$ and $y(t)$.

Lemma 1. *Let $S = [s_0, s_1]$, $T = [t_0, t_1]$.*

$$
\begin{aligned}
\chi(S_a) =& \lim_{\varepsilon\to 0}\int_{s_0}^{s_1} \mathbb{1}\{x(s)\ge a\}\bigl(-\ddot{x}(s)\bigr)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}ds \\
&+\mathbb{1}\{x(s_0)\ge a,\, \dot{x}(s_0)<0\}+\mathbb{1}\{x(s_1)\ge a,\, \dot{x}(s_1)>0\}, \\
\chi(T_b) =& \lim_{\varepsilon\to 0}\int_{t_0}^{t_1} \mathbb{1}\{y(t)\ge b\}\bigl(-\ddot{y}(t)\bigr)\frac{\mathbb{1}\{\dot{y}(t)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}dt \\
&+\mathbb{1}\{y(t_0)\ge b,\, \dot{y}(t_0)<0\}+\mathbb{1}\{y(t_1)\ge b,\, \dot{y}(t_1)>0\}.
\end{aligned}
$$

Proof. For a critical point $s^*\in\mathrm{int}(S)$,

$$
g(s^*) = \lim_{\varepsilon\to 0}\int_{\text{small region}\,(\ni s^*)} g(s)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\bigl|\ddot{x}(s)\bigr|ds
$$

holds, because by changing a variable $\dot{x}(s)=y$, we have $|\ddot{x}(s)|ds = dy$. For $\ddot{x}(s^*)>0$,

$$
\mathrm{RHS} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} g\bigl((\dot{X})^{-1}(y)\bigr)dy = g(s^*).
$$

For $\ddot{x}(s^*)<0$,

$$
\mathrm{RHS} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\varepsilon}^{-\varepsilon} g\bigl((\dot{x})^{-1}(y)\bigr)d(-y) = g(s^*).
$$

Therefore, for any function g,

$$
\sum_{s\in S^*\cap\mathrm{int}(S)} g(s) = \lim_{\varepsilon\to 0}\int_{s_0}^{s_1} g(s)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\bigl|\ddot{x}(s)\bigr|ds.
$$

Substituting $g(s)=\mathbb{1}\{x(s)\ge a\}\mathrm{sgn}\bigl(-\ddot{x}(s)\bigr)$, we have

$$
\begin{aligned}
\text{1st term of } \chi(S_a) &= \sum_{s\in S^*\cap\mathrm{int}(S)} \mathbb{1}\{x(s)\ge a\}\mathrm{sgn}\bigl(-\ddot{x}(s)\bigr) \\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1} \mathbb{1}\{x(s)\ge a\}\mathrm{sgn}\bigl(-\ddot{x}(s)\bigr)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\bigl|\ddot{x}(s)\bigr|ds \\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1} \mathbb{1}\{x(s)\ge a\}\bigl(-\ddot{x}(s)\bigr)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}ds,
\end{aligned}
$$

i.e., sgn and the absolute value symbol are canceled. □

By multiplying $\chi(S_a)$ and $\chi(T_b)$, and taking expectation, we have four terms

$$
E[\chi(S_a)\chi(T_b)] = F_1(a,b)+F_2(a,b)+F_3(a,b)+F_4(a,b),
$$

where

$$
\begin{aligned}
&F_1(a,b)\\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\int_{t_0}^{t_1} E\left[\mathbb{1}\{x(s)\ge a\}\mathbb{1}\{y(t)\ge b\}\ddot{x}(s)\ddot{y}(t)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\frac{\mathbb{1}\{\dot{y}(t)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\right]dsdt\\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\int_{t_0}^{t_1} \frac{E\big[\mathbb{1}\{x(s)\ge a,\, y(t)\ge b\}\ddot{x}(s)\ddot{y}(t)\,\mathbb{1}\{\dot{x}(s),\dot{y}(t)\in(-\varepsilon,\varepsilon)\}\big]}{E\big[\mathbb{1}\{\dot{x}(s),\dot{y}(t)\in(-\varepsilon,\varepsilon)\}\big]}\\
&\qquad\qquad\times\frac{E\big[\mathbb{1}\{\dot{x}(t),\dot{y}(t)\in(-\varepsilon,\varepsilon)\}\big]}{(2\varepsilon)^2}dsdt\\
&= \int_{s_0}^{s_1}\int_{t_0}^{t_1} E\big[\mathbb{1}\{x(s)\ge a,\, y(t)\ge b\}\ddot{x}(s)\ddot{y}(t)\,|\,(\dot{x}(s),\dot{y}(t))=0\big]\theta_{(\dot{x}(s),\dot{y}(t))}(0)dsdt
\end{aligned}
$$

with

$$
\theta_{(\dot{x}(s),\dot{y}(t))}(0)=\lim_{\varepsilon\to 0}\frac{E\big[\mathbb{1}\{\dot{x}(t),\dot{y}(t)\in(-\varepsilon,\varepsilon)\}\big]}{(2\varepsilon)^2}=\lim_{\varepsilon\to 0}\frac{\Pr(\dot{x}(s),\dot{y}(t)\in(-\varepsilon,\varepsilon))}{(2\varepsilon)^2},
$$

the density of $(\dot{x}(s),\dot{y}(t))$ evaluated at 0. Moreover, $F_1(a,b)$ is rewritten as

$$
\begin{aligned}
F_1(a,b) = \int_{s_0}^{s_1}\int_{t_0}^{t_1} E\big[\mathbb{1}\{u\ge a,\, v\ge b\}E\big[\ddot{x}(s)\ddot{y}(t)\,|\,(x(s),y(t))=(u,v),\,(\dot{x}(s),\dot{y}(t))=0\big]\\
|\,(\dot{x}(s),\dot{y}(t))=0\big]\,\theta_{(\dot{x}(s),\dot{y}(t))}(0)dsdt, \qquad (1)
\end{aligned}
$$

where (u,v) is distributed as the conditional distribution of $(x(s),y(t))=(u,v)$ given $(\dot{x}(s),\dot{y}(t))=0$.

$$
\begin{aligned}
&F_2(a,b)\\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1} E\left[\mathbb{1}\{x(s)\ge a\}\mathbb{1}\{y(t_0)\ge b,\,\dot{y}(t_0)<0\}\big(-\ddot{x}(s)\big)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\right]ds\\
&\quad+\lim_{\varepsilon\to 0}\int_{s_0}^{s_1} E\left[\mathbb{1}\{x(s)\ge a\}\mathbb{1}\{y(t_1)\ge b,\,\dot{y}(t_1)>0\}\big(-\ddot{x}(s)\big)\frac{\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}}{2\varepsilon}\right]ds\\
&= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\frac{E\big[\mathbb{1}\{x(s)\ge a,\,y(t_0)\ge b,\,\dot{y}(t_0)<0\}\big(-\ddot{x}(s)\big)\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}{E\big[\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}\\
&\qquad\qquad\times\frac{E\big[\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}{2\varepsilon}ds\\
&\quad+\lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\frac{E\big[\mathbb{1}\{x(s)\ge a,\,y(t_1)\ge b,\,\dot{y}(t_1)>0\}\big(-\ddot{x}(s)\big)\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}{E\big[\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}\\
&\qquad\qquad\times\frac{E\big[\mathbb{1}\{\dot{x}(s)\in(-\varepsilon,\varepsilon)\}\big]}{2\varepsilon}ds\\
&= \int_{s_0}^{s_1} E\big[\mathbb{1}\{x(s)\ge a,\,y(t_0)\ge b,\,\dot{y}(t_0)<0\}\big(-\ddot{x}(s)\big)\,|\,\dot{x}(s)=0\big]\theta_{\dot{x}(s)}(0)ds\\
&\quad+\int_{s_0}^{s_1} E\big[\mathbb{1}\{x(s)\ge a,\,y(t_1)\ge b,\,\dot{y}(t_1)>0\}\big(-\ddot{x}(s)\big)\,|\,\dot{x}(s)=0\big]\theta_{\dot{x}(s)}(0)ds, \qquad (2)
\end{aligned}
$$

where $\theta_{\dot{x}(s)}(0)$ is the density of $\dot{x}(s)$ evaluated at 0.

$F_3(a,b)$ is $F_2(a,b)$ with the replacement $x \leftrightarrow y$, $a \leftrightarrow b$.

The last term is

$$
\begin{aligned}
F_4(a,b) =& E[(\mathbb{1}\{x(s_0) \geq a,\, \dot{x}(s_0) < 0\} + \mathbb{1}\{x(s_1) \geq a,\, \dot{x}(s_1) > 0\}) \\
& \quad \times (\mathbb{1}\{y(t_0) \geq b,\, \dot{y}(t_0) < 0\} + \mathbb{1}\{y(t_1) \geq b,\, \dot{y}(t_1) > 0\})] \\
=& \Pr\left(x(s_0) \geq a,\, y(t_0) \geq b,\, \dot{x}(s_0) < 0,\, \dot{y}(t_0) < 0\right) \\
& + \Pr\left(x(s_0) \geq a,\, y(t_1) \geq b,\, \dot{x}(s_0) < 0,\, \dot{y}(t_1) > 0\right) \\
& + \Pr\left(x(s_1) \geq a,\, y(t_0) \geq b,\, \dot{x}(s_1) > 0,\, \dot{y}(t_0) < 0\right) \\
& + \Pr\left(x(s_1) \geq a,\, y(t_1) \geq b,\, \dot{x}(s_1) > 0,\, \dot{y}(t_1) > 0\right).
\end{aligned}
$$

2 Unit-variance bivariate Gaussian process

In the following, we assume that $(x(s), y(t)) \in \mathbb{R}^2$, $(s,t) \in S \times T$, is a zero-mean, unit-variance Gaussian process with a smooth sample path, where $S, T \in \mathbb{R}^1$ are intervals. By taking derivatives for $E[x(s)] = E[y(t)] = 0$, $E[x(s)^2] = E[y(t)^2] = 1$, we have

$$
E[x(s)\dot{x}(s)] = E[y(t)\dot{y}(t)] = 0, \quad E[x(s)\ddot{x}(s)] = -E[\dot{x}(s)^2], \quad E[y(t)\ddot{y}(t)] = -E[\dot{y}(y)^2].
$$

The cross correlation function is denoted by

$$
E[x(s)y(t)] = c(s,t).
$$

We call the stationary case when $c(s,t)$ is a function of $s-t$, i.e., $c(s,t) = c(u)$, $u = s - t$.

Assumption 1. *The maximum of $c(s,t)$ is uniquely attained at $(s,t) = (s^*, t^*) \in \mathrm{int}(S \times T)$ for the nonstationary case. For the stationary case, the maximum of $c(s,t) = c(u)$ $(u = s-t)$ is uniquely attained at $u = u^* \in \mathrm{int}\{s - t \mid (s,t) \in S \times T\}$.*

We first evaluate $F_1(a,b)$. Let s and t be fixed. Write $D_s = d/ds$, $D_t = d/dt$. Using the notations

$$
\begin{aligned}
E[D_s^i x(s) D_s^j x(s)] &= \frac{\partial^{i+j}}{\partial s^i \partial \widetilde{s}^j} E[x(s)x(\widetilde{s})]\Big|_{s=\widetilde{s}} = v_{ij}(s), \\
E[D_t^i y(t) D_t^j y(t)] &= \frac{\partial^{i+j}}{\partial t^i \partial \widetilde{t}^j} E[y(t)y(\widetilde{t})]\Big|_{t=\widetilde{t}} = w_{ij}(t),
\end{aligned}
$$

and

$$
E[D_s^i x(s) D_t^j y(t)] = \frac{\partial^{i+j}}{\partial s^i \partial t^j} E[x(s)y(t)] = \frac{\partial^{i+j}}{\partial s^i \partial t^j} c(s,t) = c_{ij}(s,t),
$$

we have the covariance matrix of $(\dot{x}(s), \dot{y}(t), x(s), y(t), \ddot{x}(s), \ddot{y}(t))$ as

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{10} & \Sigma_{12} \\ \Sigma_{01} & \Sigma_{00} & \Sigma_{02} \\ \Sigma_{21} & \Sigma_{20} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} v_{11} & c_{11} & 0 & c_{10} & v_{12} & c_{12} \\ c_{11} & w_{11} & c_{01} & 0 & c_{21} & w_{12} \\ 0 & c_{01} & 1 & c & -v_{11} & c_{02} \\ c_{10} & 0 & c & 1 & c_{20} & -w_{11} \\ v_{12} & c_{21} & -v_{11} & c_{20} & v_{22} & c_{22} \\ c_{12} & w_{12} & c_{02} & -w_{11} & c_{22} & w_{22} \end{pmatrix},$$

where $c_{ij} = c_{ij}(s,t)$, $v_{ij} = v_{ij}(s)$, $w_{ij} = w_{ij}(t)$. For the evaluation of $F_1(a,b)$ in (1), we need the following three distributions:

(i) Density of the marginal distribution of $(\dot{x}(s), \dot{y}(t))$ at $(0,0)$.

(ii) Conditional distribution of $(x(s), y(t))$ given $(\dot{x}(s), \dot{y}(t)) = (0,0)$.

(iii) Expectation of $\ddot{x}(s)\ddot{y}(t)$ under the conditional distribution given $(x(s), y(t), \dot{x}(s), \dot{y}(t)) = (u, v, 0, 0)$.

We will evaluate them in turn.

(i) The distribution of $(\dot{x}(s), \dot{y}(t))$ is $N(0, \Sigma_{11})$, and its density evaluated at $(0,0)$ is

$$\frac{1}{2\pi|\Sigma_{11}|^{\frac{1}{2}}}. \tag{3}$$

(ii) The conditional distribution of $(x(s), y(t))$ given $(\dot{x}(s), \dot{y}(t)) = (0,0)$ is

$$N_2\left(0, \Sigma_{00\cdot 1}\right), \quad \Sigma_{00\cdot 1} = \Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-1}\Sigma_{10}.$$

Its density at $(x(s), y(t)) = (u, v)$ is

$$\frac{1}{2\pi|\Sigma_{00\cdot 1}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(u,v)\Sigma_{00\cdot 1}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right\}. \tag{4}$$

(iii) The distribution of $(\ddot{x}(s), \ddot{y}(t))$ given $(x(s), y(t), \dot{x}(s), \dot{y}(t)) = (u, v, 0, 0)$ is Gaussian with mean

$$\begin{aligned} (\Sigma_{21}, \Sigma_{20})\begin{pmatrix} \Sigma_{11} & \Sigma_{10} \\ \Sigma_{01} & \Sigma_{00} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ u \\ v \end{pmatrix} &= (\Sigma_{21}, \Sigma_{20})\begin{pmatrix} * & -\Sigma_{11}^{-1}\Sigma_{10}\Sigma_{00\cdot 1}^{-1} \\ * & \Sigma_{00\cdot 1}^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ u \\ v \end{pmatrix} \\ &= (\Sigma_{20} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{10})\Sigma_{00\cdot 1}^{-1}\begin{pmatrix} u \\ v \end{pmatrix} =: H\begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

and covariance matrix

$$\Sigma_{22\cdot 10} = \Sigma_{22} - (\Sigma_{21}, \Sigma_{20})\begin{pmatrix} \Sigma_{11} & \Sigma_{10} \\ \Sigma_{01} & \Sigma_{00} \end{pmatrix}^{-1}\begin{pmatrix} \Sigma_{12} \\ \Sigma_{02} \end{pmatrix}.$$

Hence, the expectation of $\ddot{x}(s)\ddot{y}(t)$ given $(x(s), y(t), \dot{x}(s), \dot{y}(t)) = (u, v, 0, 0)$ is

$$(h_{11}u + h_{12}v)(h_{21}u + h_{22}v) + (\Sigma_{22\cdot 10})_{12}, \tag{5}$$

where

$$H = (h_{ij}) = (\Sigma_{20} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{10})\Sigma_{00\cdot 1}^{-1}.$$

Combining (i)–(iii), we can calculate the expectation of $F_1(a, b)$ in (1). The expectation of $F_2(a, b)$ in (2) can be calculated similarly.

3 Asymptotic behavior when $a, b \to \infty$

In the following, we restrict our attention to the case $a = b$. The results can be easily extended to the case $a = db$, where $d > 0$ is a constant.

Lemma 2. *For $x = (x_1, x_2) \sim N(0, K^{-1})$, $K = (k_{ij})_{2\times 2}$, as $a \to \infty$,*

$$\begin{aligned} E\big[x_i x_j \mathbb{1}\{x_1 \geq a\}\mathbb{1}\{x_2 \geq a\}\big] =& \frac{|K|^{\frac{1}{2}}}{2\pi\widetilde{k}_1\widetilde{k}_2} e^{-a^2\widetilde{k}} \Big\{1 + \frac{k_{12}}{\widetilde{k}^2}a^{-2} + O(a^{-4})\Big\}, \\ E\big[\mathbb{1}\{x_1 \geq a\}\mathbb{1}\{x_2 \geq a\}\big] =& \frac{|K|^{\frac{1}{2}}}{2\pi\widetilde{k}_1\widetilde{k}_2} e^{-a^2\widetilde{k}} \Big\{a^{-2} + O(a^{-4})\Big\}, \end{aligned}$$

where $\widetilde{k}_1 = k_{11} + k_{12}$, $\widetilde{k}_2 = k_{12} + k_{22}$, $\widetilde{k} = (k_1 + k_2)/2$, $|K| = k_{11}k_{22} - k_{12}^2$.

Proof. We follow the proof by Ruben (1964). Let

$$f(x, K) = \frac{|K|^{\frac{1}{2}}}{2\pi} e^{-\frac{1}{2}x^T K x}$$

be the density of $N_2(0, K^{-1})$. Then, by making change of variables $y = x - a\mathbb{1}$, $\mathbb{1} = (1, 1)$,

$$\begin{aligned} E\big[x_i x_j \mathbb{1}\{x_1 \geq a\}\mathbb{1}\{x_2 \geq a\}\big] =& \int_{x \geq a\mathbb{1}} x_i x_j f(x, K)dx \\ =& f(a\mathbb{1}, K)\int_{y \geq 0} (a + y_i)(a + y_j)e^{-a\mathbb{1}^T K y} e^{-\frac{1}{2}y^T K y} dy \\ =& f(a\mathbb{1}, K)\int_{y \geq 0} (a^2 + a y_i + a y_j + y_i y_j)e^{-a(\widetilde{k}_1 y_1 + \widetilde{k}_2 y_2)} e^{-\frac{1}{2}y^T K y} dy, \end{aligned} \tag{6}$$

where

$$f(a\mathbb{1}, K) = \frac{|K|^{\frac{1}{2}}}{2\pi} e^{-a^2\widetilde{k}}, \quad |K| = k_{11}k_{22} - k_{12}^2.$$

Here,

$$e^{-\frac{1}{2}y^T K y} = 1 - \frac{1}{2}(k_{11}y_1^2 + k_{22}y_2^2 + 2k_{12}y_1 y_2) + \text{4th degree in } y + \cdots,$$

and
$$\int_{y_i\ge 0} e^{-a\widetilde{k}_i y_i} y_i^m dy_i = \frac{m!}{(a\widetilde{k}_i)^{m+1}},$$
we have
$$\begin{aligned}(6) =& f(a\mathbb{1},K)\Bigg\{\frac{a^2}{(a\widetilde{k}_1)(a\widetilde{k}_2)}+\frac{a}{(a\widetilde{k}_i)^2(a\widetilde{k}_{i'})}+\frac{a}{(a\widetilde{k}_j)^2(a\widetilde{k}_{j'})}+O(a^{-4})\\ &-\frac{a^2}{2}\bigg(\frac{2!k_{11}}{(a\widetilde{k}_1)^3(a\widetilde{k}_2)}+\frac{2!k_{22}}{(a\widetilde{k}_1)(a\widetilde{k}_2)^3}+\frac{2k_{12}}{(a\widetilde{k}_1)^2(a\widetilde{k}_2)^2}+O(a^{-6})\bigg)\\ &(i' \text{ and } j' \text{ are s.t. } \{i,i'\}=\{j,j'\}=\{1,2\})\\ =& f(a\mathbb{1},K)\frac{1}{\widetilde{k}_1\widetilde{k}_2}\Bigg\{1+\bigg(\frac{1}{\widetilde{k}_i}+\frac{1}{\widetilde{k}_j}-\frac{2\widetilde{k}}{\widetilde{k}_1\widetilde{k}_2}-k_{12}\frac{\widetilde{k}_1^2\widetilde{k}_2^2-\widetilde{k}_1^2-\widetilde{k}_2^2}{\widetilde{k}_1^3\widetilde{k}_2^3}\bigg)a^{-2}+O(a^{-4})\Bigg\}.\end{aligned}$$
Similarly,
$$\begin{aligned}E\big[\mathbb{1}\{x_1\ge a\}\mathbb{1}\{x_2\ge a\}\big] =& \int_{x\ge a\mathbb{1}} f(x,K)dx\\ =& f(a\mathbb{1},K)\int_{y\ge 0} e^{-a\mathbb{1}^T K y} e^{-\frac{1}{2}y^T K y}dy\\ =& f(a\mathbb{1},K)\int_{y\ge 0} e^{-a(\widetilde{k}_1 y_1+\widetilde{k}_2 y_2)} e^{-\frac{1}{2}y^T K y}dy\\ =& f(a\mathbb{1},K)\Bigg\{\frac{1}{(a\widetilde{k}_1)(a\widetilde{k}_2)}+O(a^{-4})\Bigg\}\\ =& f(a\mathbb{1},K)\frac{1}{\widetilde{k}_1\widetilde{k}_2}\Bigg\{a^{-2}+O(a^{-4})\Bigg\}.\end{aligned}$$
□

(ii) Next we will take the expectation of (5) $\times\mathbb{1}\{u\ge a\}\mathbb{1}\{v\ge a\}$ where (u,v) is distributed as (4). We first evaluate the leading term. From Lemma 2, the result is
$$\frac{|K|^{\frac{1}{2}}h}{2\pi\widetilde{k}^2}e^{-a^2\widetilde{k}}\Bigg\{1+O(a^{-2})\Bigg\}$$
where $K=\Sigma_{00\cdot 1}^{-1}$, $h=(h_{11}+h_{12})(h_{21}+h_{22})$. By multiplying (3), we have
$$\begin{aligned}F_1(a,a)\sim& \int_S\int_T \frac{|K|^{\frac{1}{2}}}{(2\pi)^2|\Sigma_{11}|^{\frac{1}{2}}}\frac{h}{\widetilde{k}_1\widetilde{k}_2}e^{-a^2\widetilde{k}}dsdt\\ =& \int_S\int_T \frac{1}{(2\pi)^2|\Sigma_{11}|^{\frac{1}{2}}|\Sigma_{00\cdot 1}|^{\frac{1}{2}}}\frac{h}{\widetilde{k}_1\widetilde{k}_2}e^{-a^2\widetilde{k}}dsdt.\end{aligned}$$

Case I. Suppose that $\widetilde{k}(s,t)$ has a unique maximum at $(s,t)=(s^*,t^*)$. Then
$$(s,t)=(s^*,t^*) \quad\Leftrightarrow\quad c_{10}(s^*,t^*)=c_{01}(s^*,t^*)=0,$$

and at the maximum point (s^*, t^*),

$$\widetilde{k}(s^*, t^*) = \frac{1}{1 + c(s^*, t^*)} = \frac{1}{1 + \max_{s,t} c(s,t)}.$$

Denoting the Hesse matrix at (s^*, t^*) by

$$\Delta = \begin{pmatrix} \frac{\partial^2}{\partial s^2} & \frac{\partial^2}{\partial s \partial t} \\ \frac{\partial^2}{\partial s \partial t} & \frac{\partial^2}{\partial t^2} \end{pmatrix} \widetilde{k}(s,t) \Big|_{(s^*,t^*)},$$

the Laplace method yields

$$\begin{aligned} F_1(a,a) &\sim \frac{2\pi |\Delta|^{-\frac{1}{2}}}{a^2} \frac{1}{(2\pi)^2 |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{00\cdot 1}|^{\frac{1}{2}}} \frac{h}{\widetilde{k}_1 \widetilde{k}_2} e^{-a^2 \widetilde{k}} \Big|_{(s^*,t^*)} \\ &= \sqrt{\frac{(v_{11} - c_{20})(w_{11} - c_{02})}{c_{20}c_{02} - c_{11}^2}} \frac{(1+c)^2}{2\pi\sqrt{1-c^2}a^2} e^{-\frac{a^2}{1+c}} \Big|_{(s^*,t^*)} \end{aligned} \tag{7}$$

(by Mathematica).

Similarly, we can prove that

$$\begin{aligned} F_2(a,a) &\asymp a^{-2} \exp\left\{ -\frac{a^2}{1 + \max_{(s,t)\in S\times\{t_0,t_1\}} c(s,t)} \right\}, \\ F_3(a,a) &\asymp a^{-2} \exp\left\{ -\frac{a^2}{1 + \max_{(s,t)\in \{s_0,s_1\}\times T} c(s,t)} \right\}, \\ F_4(a,a) &\asymp a^{-2} \exp\left\{ -\frac{a^2}{1 + \max_{(s,t)\in \{s_0,s_1\}\times\{t_0,t_1\}} c(s,t)} \right\}, \end{aligned}$$

which are asymptotically smaller than $F_1(a,a)$ by Assumption 1.

Theorem 1. *For the nonstationary case,*

$$E[\chi(S_a)\chi(T_a)] \sim \sqrt{\frac{(v_{11} - c_{20})(w_{11} - c_{02})}{c_{20}c_{02} - c_{11}^2}} \frac{(1+c)^2}{2\pi\sqrt{1-c^2}a^2} e^{-\frac{a^2}{1+c}} \Big|_{(s^*,t^*)},$$

where $v_{11} = E[\dot{x}(s)^2]$, $w_{11} = E[\dot{y}(t)^2]$, $c_{ij} = E[x^{(i)}(s) y^{(j)}(t)]$, *and* $(s^*, t^*) = \mathrm{argmax}\, E[x(s)y(t)]$.

Case II. Suppose that $c(s,t) = c(s-t) = c(u)$ $(s - t = u)$, i.e., stationary, and $\widetilde{k}(s,t)$ has the maximum at $s - t = u = u^*$. Then,

$$u = u^* \quad \Leftrightarrow \quad c'(u^*) = 0.$$

By letting

$$\delta := \frac{\partial^2 \widetilde{k}(u)}{\partial u^2} \Big|_{u^*},$$

the Laplace method yields

$$F_1(a,a) \sim \mu(S\cap T)\frac{\sqrt{2\pi\delta^{-1}}}{a}\frac{1}{(2\pi)^2|\Sigma_{11}|^{\frac{1}{2}}|\Sigma_{00\cdot 1}|^{\frac{1}{2}}}\frac{h}{\widetilde{k}^2}e^{-a^2\widetilde{k}}\Big|_{t-s=u^*}$$
$$= \mu(S\cap T)\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{(v_{11}-c'')(w_{11}-c'')}{-c''}}\frac{1+c}{\sqrt{1-c^2}a}e^{-\frac{1}{c+1}a^2}\Big|_{t-s=u^*}.$$

Theorem 2. *For the stationary case,*

$$E[\chi(S_a)\chi(T_a)] \sim \mu(S\cap T)\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{(v_{11}-c'')(w_{11}-c'')}{-c''}}\frac{1+c}{\sqrt{1-c^2}a}e^{-\frac{1}{c+1}a^2}\Big|_{t-s=u^*}.$$

4 Comparisons to the existing results

Anshin (2006) derived the corresponding results when $c_{11}(s^*,t^*)=0$:

$$\sqrt{\frac{v_{11}w_{11}}{c_{20}c_{02}}}\frac{(1+c)^2}{2\pi\sqrt{1-c^2}a^2}e^{-\frac{a^2}{1+c}}\Big|_{(s^*,t^*)}. \tag{8}$$

In the numerator, c_{20} and c_{02} are missing.

On the other hand, from Zhou and Xiao (2017) noting that $H_2 = 1/\sqrt{\pi}$ (page 31 of Piterberg (1996)), by substituting $N=1$, $\alpha_i=2$, $c_i=1/2$, the corresponding formula is

$$\mu(S\cap T)\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{v_{11}w_{11}}{-c''}}\frac{1+c}{\sqrt{1-c^2}a}e^{-\frac{1}{c+1}a^2}.$$

Two c'' are missing again.

Here is a counter example to Anshin (2006): Using a Gaussian vector $\xi=(\xi_1,\xi_2,\xi_3)^T\sim N_3(0,I)$, define $x(s)=\xi^T\varphi_1(s)$, $y(t)=\xi^T\varphi_2(t)$ with

$$\varphi_x(s)=(s,d+Rs^2,\sqrt{1-s^2-(d+Rs^2)^2})^T,\quad \varphi_y(t)=(t,-d-Rt^2,\sqrt{1-t^2-(d+Rt^2)^2})^T,$$

where $T=S=[-\varepsilon,\varepsilon]$, a small interval including the origin. Then $x(s)$, $y(t)$ are nonstationary Gaussian processes with zero-mean, unit-variance, and $c(s,t)=\varphi_x(s)^T\varphi_y(s)$ has its maximum $1-2d^2$ at $(s^*,t^*)=(0,0)$. By simple calculations, we have $v_{11}=w_{11}=1$, $c_{20}=c_{02}=-4dR-1$, $c_{11}=1$. As $R\to\infty$, the multipliers in (7) and (8) are

$$\sqrt{\frac{(v_{11}-c_{20})(w_{11}-c_{02})}{c_{20}c_{02}-c_{11}^2}}\to 1,\quad \sqrt{\frac{v_{11}w_{11}}{c_{20}c_{02}-c_{11}^2}}\to 0,$$

respectively. Noting that the resulting probability should be bounded below by

$$P\Big(\sup x(s)\ge a,\,\sup y(t)\ge a\Big)\ge P(x(0)\ge a,\,y(0)\ge a)\sim\frac{(1+c)^2}{2\pi\sqrt{1-c^2}a^2}e^{-\frac{a^2}{1+c}},$$

the former formula (7) is consistent, and the latter formula (8) is a contradiction.

References

1. Anshin, A. B. (2006). On the probability of simultaneous extremes of two Gaussian nonstationary processes, *Theory Probab. Appl.*, **50** (3), 353–366.
2. Piterbarg, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, Translations of Mathematical Monographs **148**, AMS.
3. Ruben, H. (1964). An asymptotic expansion for the multivariate normal distribution and Mills' ratio, *Journal of Research of the National Bureau of Standards, B. Mathematics and Mathematical Physics*, **68B** (1). `https://nvlpubs.nist.gov/nistpubs/jres/68B/jresv68Bn1p3_A1b.pdf`
4. Zhou, Y. and Xiao, Y. (2017). Tail asymptotics for the extremes of bivariate Gaussian random fields, *Bernoulli*, **23** (3), 1566–1598.